IMPLEMENTATION VIA APPROVAL MECHANISMS*

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ABSTRACT. We focus on the single-peaked domain and study the class of Generalized Approval Mechanisms (*GAMs*): First, players simultaneously select subsets of the outcome space and scores are assigned to each alternative; and, then, a given quantile of the induced score distribution is implemented. Our main finding is that essentially for every Nash-implementable welfare optimum –including the Condorcet winner alternative– there exists a *GAM* that Nash-implements it. Importantly, the *GAM* that Nash-implements the Condorcet winner alternative is the first simple simultaneous game with this feature in the literature.

KEYWORDS. Nash Implementation, Strategy-proofness, Approval Voting, Single-Peakedness, Condorcet winner.

JEL CLASSIFICATION. C9, D71, D78, H41.

1. INTRODUCTION

In the single-peaked domain, the Nash-implementable welfare optima, practically, coincide with the outcomes of Generalized Median Rules (*GMRs*).[1] In simple terms, the outcome of a *GMR* is the median of a set of points that consists of: a) the voters' ideal policies and b) some exogenous values also known as phantoms. As proved by Moulin [1980] *GMRs* are the unique social choice rules that satisfy efficiency and strategy-proofness, while Berga and Moreno [2009] established that strategy-proof rules which are "not too bizarre" (in the context of Sprumont [1995])[2] are the only implementable ones.

However, one should note that the direct revelation game of each *GMR* need not lead to the same outcome as the *GMR* itself. In this respect, the direct revelation games of *GMRs* share a common feature with other strategy-proof mechanisms: They admit a large multiplicity of Nash equilibria, some of which produce different outcomes (see Saijo et al. [2007]). For instance, the direct revelation game triggered by the pure median rule –whose outcome is the Condorcet winner alternative– exhibits a large set of equilibria: As long as every player

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[1] In the present paper's context, a welfare optimum is the outcome of social choice rule (Maskin [1999]), the set of alternatives is $A = [0,1]$ and the set of possible preference relations consists of the single-peaked ones in A.

[2] That is, restricting attention to anonymous rules that implement each of the alternatives for at least one preference profile.

announces the same alternative x, this constitutes an equilibrium with outcome x since no unilateral deviation affects the median choice.[3] This leads to the following conclusion: The direct revelation game of a *GMR* does not Nash-implement the *GMR* (see Repullo [1985] for similar results).[4]

So how do we Nash-implement *GMRs* in a simple manner? Yamamura and Kawasaki [2013] propose the class of averaging mechanisms. Each player announces an alternative and a monotonic transformation of the average alternative is implemented. The equilibrium outcome coincides with the outcome of a *GMR* with an important restriction: All phantoms must be interior, which prevents, among others, the implementation of the Condorcet winner alternative. Moreover, Gershkov et al. [2016] have recently shown that sequential quota mechanisms can also implement *GMRs*.[5] Indeed, being able to implement *GMRs* by the means of simple sequential games is very important, but ideally one would like to be able to do the same using simple simultaneous games as well.

In this paper, we design the class of Generalized Approval Mechanisms (*GAMs*). These mechanisms are quite easy to describe and belong to the class of simultaneous voting games. First, players select subsets of the outcome space and scores are assigned to each alternative (hence, Approval). Given a subset of alternatives, two different *GAMs* may assign different positive scores to the same approved alternative (hence, Generalized). Then, a given quantile of the score distribution induced by the players' choices is implemented. Our main finding is that *every generic*[6] *GMR –including the Condorcet winner alternative– can be Nash-implemented by some GAM.* We explain how to derive a *GAM* for each *GMR* and we explicitly design the one that implements the Condorcet winner alternative, also known as the pure median rule. To our knowledge, this is the first simple simultaneous game that implements the Condorcet winner alternative and arguably this finding is of interest on its own. The equilibrium strategies of most players[7] take an easy "I approve every alternative at most (least) as large as the implemented alternative" form. In fact, every player with a preferred alternative to the left (right) of the implemented one approves the implemented alternative and all the alternatives to its left (right). That is, *GAMs* not only Nash-implement *GMRs*, but also promote sincerity and agreement, in the sense that most players include both their ideal policies and the implemented outcome in their approval sets.

Naturally, the present analysis relates to the wider Approval voting literature. Approval voting has been studied since Weber [1995] and Brams and Fishburn [1983], and has been shown to exhibit interesting properties in a variety of contexts: For example, it improves the quality of decisions in common value problems compared to plurality rule (Bouton and

[3]Experimental evidence shows that strategy-proof mechanisms need not lead a large share of the agents to reveal their true type (see Attiyeh et al. [2000], Kawagoe and Mori [2001], Kagel and Levin [1993] and Cason et al. [2006] among others).

[4]A game/mechanism Nash-implements a social choice rule if it admits a unique equilibrium outcome which coincides with the outcome of the social choice rule (see Maskin [1999]).

[5]More precisely, their sequential mechanisms are obtained by modifying a sequential voting scheme suggested by Bowen [1943].

[6]We consider that a *GMR* is generic if its interior phantoms –if any– are all non-identical.

[7]If a player's peak coincides with the equilibrium outcome, then this player may be employing a different kind of strategy.

Castanheira [2012]) and leads to the sincere revelation of preferences in certain private value settings (see Laslier [2009], Laslier and Sanver [2010] and Núñez [2014]). As we show, in the single-peaked domain Approval voting can additionally help a society reach, essentially, every feasible welfare optimum.

In what follows we describe the setting (section 2) and present an example (section 3). Then we provide our formal results and explain how to implement the Condorcet winner through a *GAM* (section 4).

2. The setting

2.1. **Basic concepts and definitions.** Let $A := [0,1]$ denote the set of alternatives and $N := \{1,\ldots,n\}$ the set of players with $n \geq 2$. Let U be the set of single-peaked preferences. Each player i has utility function u_i in U with $u_i(x)$ the utility of player i when $x \in A$ is implemented. Each player i has a unique peak, t_i, so that $u_i(x') < u_i(x'')$ when $x' < x'' \leq t_i$ and when $t_i \leq x'' < x'$.[8] We let $t = (t_1,\ldots,t_n)$ stand for a peak profile and $u = (u_1,\ldots,u_n) \in \mathscr{U} := \prod_{j=1}^{n} U$. A social choice function is a function $f : \mathscr{U} \to A$ that associates every $u \in \mathscr{U}$ with a unique alternative $f(u)$ in A. For any finite collection of *points* $x_1,\ldots,x_s$ in $[0,1]$, we let $m(x_1,\ldots,x_s)$ denote their median: $m(x_1,\ldots,x_s)$ is the smallest number in $\{x_1,\ldots,x_s\}$ which satisfies $\frac{1}{s}\#\{x_i \mid x_i \leq m(x_1,\ldots,x_s)\} \geq \frac{1}{2}$ and $\frac{1}{s}\#\{x_i \mid x_i \geq m(x_1,\ldots,x_s)\} \geq \frac{1}{2}$. A social choice function is a generalized median rule (*GMR*) if there is some collection of points $p_1,\ldots,p_{n-1}$ in $[0,1]$ such that, for each $u \in \mathscr{U}$, $f(u) = m(t,p_1,\ldots,p_{n-1})$. We refer to $p_1,\ldots,p_{n-1}$ as the phantoms of the *GMR*. A *GMR* is considered to be generic if its interior phantoms –if any– are non-identical.

A mechanism is a function $\theta : S \to A$ that assigns to every $s \in S$, a unique element $\theta(s)$ in A, where $S := \prod_{i=1}^{n} S_i$ and S_i is the strategy space of player i. Given a mechanism $\theta : S \to A$, the strategy profile $s \in S$ is a Nash equilibrium of θ at $u \in \mathscr{U}$, if $u_i(\theta(s_i,s_{-i})) \geq u_i(\theta(s_i',s_{-i})$ for all $i \in N$ and any $s_i' \in S_i$. Let $N^{\theta}(u)$ be the set of Nash equilibria of θ at u. The mechanism θ implements the social choice function f in Nash equilibria if for each $u \in \mathscr{U}$, a) there exists $s \in N^{\theta}(u)$ such that $\theta(s) = f(u)$ and b) for any $s \in N^{\theta}(u)$, $\theta(s) = f(u)$.

2.2. **Generalized Approval Mechanisms.** We let $\mathscr{B}$ denote the collection of closed intervals of A.[9] A *GAM* is a mechanism $\theta : \mathscr{B}^n \to A$ which requires each player to play simultaneously a strategy in $\mathscr{B}$ and determines for each strategy profile some alternative in A. For each $b_i \in \mathscr{B}$, we write $\underline{b_i} = \min b_i$ and $\overline{b_i} = \max b_i$. The set $\mathscr{B}$ includes elements of different dimensions: singletons and positive length intervals. Since each b_i is a convex set, its dimension is well-defined so that for each approval profile $b = (b_i, b_{-i})$, we let $\dim(b) = \max_{i \in N} \dim(b_i)$. The set of zero-dimensional and one-dimensional strategies are respectively labeled by $\mathscr{B}_0$ and $\mathscr{B}_1$ with $\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$. Similarly, $\mathscr{B}_0^n$ denotes the set of profiles in which every player plays a singleton and $\mathscr{B}_1^n$ the set of profiles such that at least one player plays a one-dimensional strategy.

[8]For simplicity, we assume that $t_i \neq t_j$ for any $i,j \in N$. Our results are not affected when relaxing this constraint.
[9]This assumption can be relaxed by allowing any finite union of closed and convex subsets of A to be the set of pure strategies. Relaxing it however would imply more cumbersome notation and proofs since then two strategies that differ by a zero-measure set can have equivalent consequences.

In order to state a precise definition of a *GAM*, we let $\eta : \mathbb{R} \to \mathbb{R}$ be a differentiable and strictly increasing function with $\eta(0) = 0$ and $\eta(1) = 1$ and q a non-negative real number. We assume that when player i submits the interval b_i, he is endowed with a weight of $q + \eta(\overline{b_i}) - \eta(\underline{b_i})$ to be distributed over b_i. More precisely, if $\dim(b_i) = 1$, then the strategy b_i assigns an individual score of $s_x(b_i, q, \eta)$ to each $x \in [0,1]$ as follows:

$$s_x(b_i, q, \eta) = \frac{q}{\overline{b_i} - \underline{b_i}} + \eta'(x) \text{ for any } x \in b_i \text{ and } s_x(b_i, q, \eta) = 0 \text{ otherwise,}$$

so that $s(b_i, q, \eta) = \int_0^1 s_x(b_i, q, \eta)dx$ equals $q + \eta(\overline{b_i}) - \eta(\underline{b_i})$ as defined. On the contrary, when $\dim(b_i) < \dim(b)$, b_i is a singleton and some player announces a one-dimensional interval, we let $s_x(b_i, q, \eta) = 0$ for every $x \in [0,1]$ so that his strategy is not taken into account.

Collectively, each profile b in $\mathscr{B}_1^n$ assigns to each alternative x a score of $s_x(b, q, \eta)$ with $s_x(b, q, \eta) = \sum_{i=1}^n s_x(b_i, q, \eta)$. Hence, the score distribution is the function $\phi_{q,\eta} : \mathscr{B}_1^n \times [0,1] \to [0,1]$ such that

$$\phi_{q,\eta}(b, z) = \int_0^z \frac{s_x(b, q, \eta)}{\sum_{i=1}^n s(b_i, q, \eta)} dx.$$

A *GAM* $\theta_{\alpha,q,\eta}$ associates any profile $b \in \mathscr{B}_0^n$ with $\theta_{\alpha,q,\eta}(b) = m(b_1, b_2, \ldots, b_n)$ [10] and any profile $b \in \mathscr{B}_1^n$ with

$$\theta_{\alpha,q,\eta}(b) = \min\{x \in [0,1] \mid \phi_{q,\eta}(b, \theta_{\alpha,q,\eta}(b)) = \alpha\}, \text{ where } \alpha \in (0,1).$$

In other words, a *GAM* selects the α-quantile of the distribution endogenously generated by b given q and η when at least some player announces a positive length interval; otherwise, it selects the median of the announced singletons. In what follows, we write θ rather than $\theta_{\alpha,q,\eta}$.

The initial step is to show that any *GAM* is well-defined.

Lemma 1. *For any admissible q, α and η, the associated GAM is well-defined.*

Proof. Note first that for any $b \in \mathscr{B}_1^n$, $\frac{s_x(b,q,\eta)}{\sum_{i=1}^n s(b_i,q,\eta)} \geq 0$ for every $x \in [0,1]$. It suffices to show that its integral over $[0,1]$ equals 1, which is equivalent to $\phi_{q,\eta}(b, 1) = 1$. But this is satisfied since $\int_0^1 s_x(b, q, \eta)dx = \sum_{i=1}^n s(b_i, q, \eta)$. **Q.E.D.**

For each *GAM* θ, we let the points $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ denote the phantoms of θ, with $\kappa_j = \max\{0, \min\{1, \eta^{-1}(\gamma_j)\}\}$ and $\gamma_j = \frac{\alpha(nq+j)-(n-j)q}{(n-j)-\alpha(n-2j)}$, for each $j \in \{1, \ldots, n-1\}$. Note that if $\eta^{-1}(\gamma_j) \in (0,1)$, then $\kappa_j = \eta^{-1}(\gamma_j)$. Moreover, by the means of standard algebraic manipulations, one can show that that for each $j \in \{1, \ldots, n-1\}$, each $q \in \mathbb{R}^+$, and each $\alpha \in (0,1)$, $\gamma_j < \gamma_{j+1}$. The previous inequality, combined with η being differentiable and strictly increasing, implies that $0 \leq \kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_{n-1} \leq 1$, and that as long as $\kappa_i, \kappa_j \in (0,1)$, $\kappa_i \neq \kappa_j$.

For each $j \in \{1, \ldots, n-1\}$ and each $x \in [0,1]$, we let $\mathscr{B}(j, x) := \left\{b \in \mathscr{B}^n \mid \#\{b_i = [0,x]\} = n - j \text{ and } \#\{b_i = [x,1]\} = j\right\}$. Any profile b in $\mathscr{B}(j,x)$ has $n-j$ players playing $[0,x]$ and j players playing $[x,1]$. Note that $\theta(b) = \theta(b')$ if $b, b' \in \mathscr{B}(j,x)$.

[10] This assumption is made for the sake of completeness. We could use any alternative $\lambda : [0,1]^n \to [0,1]$ rather than using the median of the singletons without affecting the results.

Lemma 2. *If θ is a GAM, $j \in \{1, \ldots, n-1\}$, and $b \in \mathscr{B}(j,x)$ with $x \in (0,1)$ then a) $\theta(b) = x$ if and only if $x = \kappa_j$; b) $\theta(b) > x$ if and only if $\kappa_j > x$; and c) $\theta(b) < x$ if and only if $\kappa_j < x$.*

Proof. For each $j \in \{1, \ldots, n-1\}$, each $x \in (0,1)$ and any $b \in \mathscr{B}(j,x)$, the distribution $\phi(b,x)$ satisfies $\phi(b,x) = \frac{(n-j)\eta(x)+(n-j)q}{qn+(n-j)\eta(x)+j(1-\eta(x))}$. Therefore, since η is differentiable and strictly increasing in $(0,1)$, $\frac{\partial}{\partial x}\phi(b,x) > 0$ for each $x \in (0,1)$. This implies that, if $\phi(b,x) = \alpha$ for some $x \in (0,1)$, then x is unique and $\phi(b,x) = \frac{(n-j)\eta(x)+(n-j)q}{qn+(n-j)\eta(x)+j(1-\eta(x))} = \alpha \Leftrightarrow x = \eta^{-1}(\gamma_j) = \kappa_j$, which proves *a*). Moreover, each $b \in \mathscr{B}(j,x)$ with $x < \kappa_j$ satisfies $\phi(b,x) < \alpha$ since $\phi(b,x)$ is strictly increasing in x, and thus $\theta(b) > x$. Similarly, each $b \in \mathscr{B}(j,x)$ with $\theta(b) > x$ is such that $\phi(b,x) < \alpha$, which implies $x < \kappa_j$ and proves *b*). A symmetric argument proves *c*) concluding the proof. **Q.E.D.**

Notice that by the definition of a *GAM* –in specific by the fact that $\alpha \in (0,1)$– it follows that $\theta(b) > x$ when $b \in \mathscr{B}(n,x)$ and $\theta(b) < x$ when $b \in \mathscr{B}(0,x)$ for every $x \in [0,1]$.

3. An Example: the Median Approval mechanism

In this section we present an example that illustrates how a specific *GAM* works for a simple class of preference profiles. We consider a society composed of three individuals with peaks such that $0 = t_1 < t_2 < t_3 = 1$. The Approval mechanisms that we consider throughout have the following common structure: a) Every player simultaneously and independently announces an interval $b_i \in \mathscr{B}$, b) these intervals generate a score distribution, and c) the mechanism implements $\theta(b)$ which equals some quantile of the score distribution such as the median. The Approval mechanisms differ in how this distribution is generated and in the quantile of the distribution that is implemented.

While the general structure is discussed in the rest of the paper, we stick here to the simplest interesting score assignment process: That is, we assume that when player i submits the interval b_i, he assigns an individual score $s_x(b_i)$ to each $x \in [0,1]$ as follows:

$$s_x(b_i) = 1 \text{ for any } x \in b_i \text{ and } s_x(b_i) = 0 \text{ otherwise.}$$

Collectively, each strategy profile b assigns a score of $s_x(b)$ to each alternative x with $s_x(b) = \sum_{i=1}^n s_x(b_i)$. If at least one player submits a positive length interval, the distribution is the function $\phi : \mathscr{B}_1^n \times [0,1] \to [0,1]$ such that

$$\phi(b,z) = \int_0^z \frac{s_x(b)}{\sum_{i=1}^n (\overline{b_i} - \underline{b_i})} dx.$$

The Median Approval mechanism associates any profile b with the median $\theta(b)$ of the score distribution (when ϕ is continuous, $\phi(b,\theta(b)) = \frac{1}{2}$, while when all players announce a singleton, $\theta(b)$ corresponds to the median of these singletons).

We first notice that for any profile b with $\theta(b) \neq t_i$ and $b_i \in \mathscr{B}_0$, player i can effectively move the median of the score distribution closer to her peak, $t_i \in (0,1)$, by submitting a sufficiently small –but non-degenerate– interval containing t_i. Hence, in equilibrium it must be the case that an individual whose peak does not coincide with the outcome uses a one-dimensional strategy and, in particular, he uses $[0,\theta(b)]$ if $t_i < \theta(b)$ and $[\theta(b),1]$ if $t_i > \theta(b)$. This is so because placing weight to alternatives to the left of the implemented one shifts the implemented alternative to the left and vice versa.

Note that for the three players example that we consider, $\theta([0,x],[0,x],[x,1]) = \frac{1-x}{2}$ if $x \le \frac{1}{3}$ and $\theta([0,x],[0,x],[x,1]) = \frac{1+x}{4}$ if $x \ge \frac{1}{3}$. Similarly, $\theta([0,x],[x,1],[x,1]) = \frac{2+x}{4}$ if $x \le \frac{2}{3}$ and $\theta([0,x],[x,1],[x,1]) = \frac{2-x}{2}$ if $x \ge \frac{2}{3}$. Therefore, $\theta([0,x],[0,x],[x,1]) = x$ if and only if $x = \kappa_1 = \frac{1}{3}$ and $\theta([0,x],[x,1],[x,1]) = x$ if and only if $x = \kappa_2 = \frac{2}{3}$. In other words, when $n = 3$, the phantoms of the Median Approval mechanism are $\kappa_1 = \frac{1}{3}$ and $\kappa_2 = \frac{2}{3}$.

The previous arguments suggest that: a) when $t_2 < \frac{1}{3}$ the unique equilibrium is $([0,\frac{1}{3}],[0,\frac{1}{3}],[\frac{1}{3},1])$ with outcome $\frac{1}{3}$ and b) when $t_2 > \frac{2}{3}$ the unique equilibrium is $([0,\frac{1}{3}],[\frac{1}{3},1],[\frac{1}{3},1])$ with outcome $\frac{2}{3}$. But what happens when $t_2 \in [\frac{1}{3},\frac{2}{3}]$? Then, in any equilibrium b, player 1 still uses $[0,\theta(b)]$ and player 3 still uses $[\theta(b),1]$, but player 2 can use a different kind of strategy and have his peak implemented. Indeed, when, for example, $t_2 \in [\frac{1}{3},\frac{1}{2}]$ an equilibrium can be such that $\theta([0,t_2],[0,4t_2-1],[t_2,1]) = t_2$. In these cases the equilibrium need not be unique, as the median player has many best responses, but the equilibrium outcome is unique and coincides with the peak of the median player. In Figure 1 we present the unique equilibrium outcome of the Median Approval mechanism for all the preference profiles that we considered here. In Figure 2 we present the scores assigned to each alternative, $s_x(b)$, in an equilibrium of the form $\theta([0,t_2],[0,4t_2-1],[t_2,1]) = t_2$.

[Insert Figure 1 and Figure 2 about here]

4. Formal Analysis

We prove first how best replies are under a *GAM* (Lemma 3), then prove that a *GAM* Nash-implements a *GMR* (Proposition 1) and, finally, establish that for every generic *GMR* there exists a *GAM* that Nash-implements it (Theorem 1).

Next, we assert that if a player whose peak lies to the left (right) of the outcome uses a best response, then he approves of all the alternatives to the left (right) of the implemented outcome.

Lemma 3. *If θ is a GAM, and $b = (b_i, b_{-i}) \in \mathscr{B}^n$ with $t_i < \theta(b)$ ($t_i > \theta(b)$), then b_i is a best response to b_{-i} if and only if $b_i = [0,\theta(b)]$ ($b_i = [\theta(b),1]$).*

Proof. We only provide a proof for the case in which $t_i < \theta(b)$ since the proof for $t_i > \theta(b)$ is symmetric. We first consider a strategy profile $b = (b_i, b_{-i})$ with $t_i < \theta(b)$ and $b_i \neq [0,\theta(b)]$ and argue that b_i cannot be a best response of player i; and then we consider a strategy profile b with $t_i < \theta(b)$ and $b_i = [0,\theta(b)]$ and argue that b_i is a best response of player i.

When $t_i < \theta(b)$ and $b_i \neq [0,\theta(b)]$ there are three possibilities regarding b_i: a) $\theta(b) > \overline{b_i}$, b) $\theta(b) < \underline{b_i}$; and c) $\theta(b) \in b_i$. If $\theta(b) > \overline{b_i}$ and dim(b)=0 then i can submit a sufficiently small –but non-degenerate– interval centered at t_i and bring the implemented outcome arbitrarily closer to his peak.[11] If $\theta(b) > \overline{b_i}$, dim($b$)=1 and dim($b_i$)=0 then i can deviate to $[t_i,\theta(b)]$ and induce $\phi(([t_i,\theta(b)],b_{-i}),\theta(b)) > \phi(b,\theta(b))$ and $\phi(([t_i,\theta(b)],b_{-i}),t_i) \le \phi(b,t_i)$; and hence bring the implemented outcome closer to her peak. If $\theta(b) > \overline{b_i}$ and dim(b_i)=1 then, there exists $\beta \in (0,\theta(b))$ such that $\theta(b) = \theta([\beta,\theta(b)],b_{-i})$. This is so because the outcome of a *GAM* does

[11] When $t_i = 0$, the player can submit an interval $[0,\varepsilon]$, with a sufficiently small $\varepsilon > 0$, to the described effect (the case of $t_i = 1$ is symmetric).

not depend on the specific interval that one submits when this interval contains outcomes only to the left (right) of the implemented one, but only on the total weight assigned to policies on the left (right) of the implemented outcome. We assume that i deviates to such a strategy, $[\beta, \theta(b)]$, that delivers the same outcome as b_i.[12] After this intermediate step, we simply consider marginal changes in β. Indeed, one can show that $\frac{\partial}{\partial\beta}\phi(([\beta, \theta(b)], b_{-i}), \theta(b)) < 0$ which means that the implemented outcome $\theta([\beta, \theta(b)], b_{-i})$ continuously decreases when β decreases; and this clearly improves the payoff of player i. That is, b_i cannot be a best response of player i. Case b) admits a completely symmetric proof. Case c) is actually simpler since it is such that $\underline{b_i} \leq \theta(b) \leq \overline{b_i}$, so one can consider directly marginal changes of $\underline{b_i}$ and/or $\overline{b_i}$ without the need for the described intermediate step.

Now consider that $t_i < \theta(b)$ and $b_i = [0, \theta(b)]$, and that there exists b_i' such that $u_i(\theta(b_i', b_{-i})) > u_i(\theta(b))$. If $b_i' \not\subset b_i$ then $\phi((b_i', b_{-i}), \theta(b)) < \phi(b, \theta(b))$ and hence $\theta(b_i', b_{-i}) > \theta(b)$. If $b_i' \subset b_i$ then $b_i' = [0, \beta]$ for some $\beta > \theta(b)$. One can show that $\frac{\partial}{\partial\beta}\phi(([0, \beta], b_{-i}), \theta(b)) < 0$ when $\beta > \theta(b)$. That is, a transition from b_i to b_i' will induce $\phi((b_i', b_{-i}), \theta(b)) < \phi(b, \theta(b))$ and hence $\theta(b_i', b_{-i}) > \theta(b)$. In both cases the assumption that $b_i = [0, \theta(b)]$ is not a best response is contradicted and this concludes the argument. **Q.E.D.**

Next we establish that a *GAM* implements a *GMR* in Nash equilibria.

Proposition 1. *If the mechanism $\theta : \mathscr{B}^n \to A$ is a Generalized Approval Mechanism (GAM) then:*

a) there is an equilibrium in pure strategies for every admissible preference profile; and

b) in every equilibrium b of θ we have $\theta(b) = m(t_1, t_2, \ldots, t_n, \kappa_1, \ldots, \kappa_{n-1})$.

Proof. Take some *GAM* mechanism $\theta : \mathscr{B}^n \to A$. The proof first establishes the existence of an equilibrium (**Step A.**) and then fully characterizes the unique equilibrium outcome (**Step B.**). For short, we write (t, κ) rather than $(t_1, t_2, \ldots, t_n, \kappa_1, \ldots, \kappa_{n-1})$.

Step A.: There is some equilibrium b of θ with $\theta(b) = m(t, \kappa)$.

Step A. is divided into two cases: There is either no t_h with $t_h = m(t, \kappa)$ (Step A.I.), or there is a t_h with $t_h = m(t, \kappa)$ (Step A.II.).

Step A.I. There is no t_h with $t_h = m(t, \kappa)$. Since there is no t_h with $t_h = m(t, \kappa)$, there must exist $j \in \{1, \ldots, n-1\}$ such that $\kappa_j = m(t, \kappa)$. Therefore, the number of elements located below and above κ_j in (t, κ) is equal to $n-1$, which is equivalent to:

$$\underbrace{\#\{i \in N \mid t_i < \kappa_j\} + (j-1)}_{\text{elements strictly lower than } \kappa_j} = \underbrace{\#\{i \in N \mid t_i > \kappa_j\} + (n-j-1)}_{\text{elements strictly higher than } \kappa_j} = n-1.$$

The previous equalities imply that $\#\{i \in N \mid t_i < \kappa_j\} = n - j$ and $\#\{i \in N \mid t_i > \kappa_j\} = j$. Let $b \in \mathscr{B}(j, \kappa_j)$ be such that:

$$b_i := \begin{cases} [0, \kappa_j] & \text{if } t_i < \kappa_j, \\ [\kappa_j, 1] & \text{if } t_i > \kappa_j. \end{cases}$$

[12] In specific, β is uniquely defined by $\eta(\theta(b)) - \eta(\beta) = \eta(\overline{b_i}) - \eta(\underline{b_i})$.

By Lemma 2, $\theta(b) = \kappa_j$ and therefore $\theta(b) = m(t,\kappa)$. Since every player is playing a best response as defined in Lemma 3, b is an equilibrium of the game and this concludes Step A.I.

Step A.II. There is some t_h with $t_h = m(t,\kappa)$. If there exists $j \in \{1,\ldots,n-1\}$ such that $\kappa_j = t_h$, then either $j = n-h$ or $j = n-h+1$. Using the same line of reasoning as in A.I., one can show that: a) when $j = n-h$, any $b \in \mathscr{B}(n-h, t_h)$ is an equilibrium with $\theta(b) = t_h$ and b) when $j = n-h+1$, any $b \in \mathscr{B}(n-h+1, t_h)$ is an equilibrium with $\theta(b) = t_h$.

If $t_h = m(t,\kappa)$ and $t_h \neq \kappa_j$, there are $n-1$ values strictly smaller than t_h in (t,κ). There are essentially two cases here: a) $t_h \in (\kappa_1, \kappa_{n-1})$ and b) $t_h < \kappa_1$ (the proof for the case $t_h > \kappa_{n-1}$ is symmetric). Below, we consider both cases in turn.

a) Choose j, such that $1 \leq j \leq n-2$, with $\kappa_j < t_h = m(t,\kappa) < \kappa_{j+1}$. Moreover $\#\{\kappa_l \mid \kappa_l < t_h\} = j$ and $\#\{i \in N \mid t_i < t_h\} = h-1$ so that: $j+h-1 = n-1 \Longrightarrow j = n-h$. Therefore, $\kappa_{n-h} < t_h < \kappa_{n-h+1}$.

For each strategy $c^* \in \mathscr{B}$, we let $b = (c^*, b_{-h})$ denote a strategy profile with:

$$b_i = \begin{cases} [0,t_h] & \text{if } t_i < t_h, \\ c^* & \text{if } t_i = t_h, \\ [t_h,1] & \text{if } t_i > t_h. \end{cases}$$

Our objective is to prove that there is some c^* such that $\theta(b) = t_h$ and b is an equilibrium.

By Lemma 2, it follows that if $\kappa_{n-h} \in (0,1)$, $\theta(b') = \kappa_{n-h} < t_h$ for any $b' \in \mathscr{B}(n-h, \kappa_{n-h})$ and, if $\kappa_{n-h} = 0$, $\theta(b') < t_h$ for any $b' \in \mathscr{B}(n-h, t_h)$. Again, due to Lemma 2, if $\kappa_{n-h+1} \in (0,1)$, $\theta(b') = \kappa_{n-h+1} > t_h$ for any $b' \in \mathscr{B}(n-h+1, \kappa_{n-h+1})$ and if $\kappa_{n-h+1} = 1$, $\theta(b') > t_h$ for any $b' \in \mathscr{B}(n-h+1, t_h)$. Hence, it follows that $\theta([0,t_h], b_{-h}) < t_h$ and $\theta([t_h,1], b_{-h}) > t_h$, so that there exists some c^* with $\theta(b) = t_h$. This is so because when the rest of the players behave according to b_{-h}, h can smoothly deviate from $[0,t_h]$ to $[t_h,1]$ –first, continuously increase the right bound of his interval up to 1, and, then, continuously increase the left bound of his interval up to his peak– and induce a continuous change of the implemented policy from $\theta([0,t_h], b_{-h})$ to $\theta([t_h,1], b_{-h})$.

In order to prove that $b = (c^*, b_{-h})$ with $\theta(b) = t_h$ is an equilibrium, suppose by contradiction that there exists some $i \in N$ with a profitable deviation b_i'. Yet, as proved by Lemma 3, any player with a peak different than t_h is playing a best response in b. Moreover, the player with peak t_h is also playing a best response since $\theta(b) = t_h$. Therefore, b must be an equilibrium concluding a) in Step A.

b) In this case, $t_h = m(t,\kappa) < \kappa_1$, and hence, $h = n$. Therefore, in any equilibrium b, the $n-1$ players with peak strictly lower than t_n play $[0,t_n]$. Moreover, $\theta([0,t_n], b_{-n}) < t_n$, since for any $b \in \mathscr{B}(0,x)$, $\theta(b) < x$ for every $x \in (0,1)$; and $\theta([t_n,1], b_{-n}) > t_n$, since for any $b \in \mathscr{B}(1,x)$, $\theta(b) > x$ if $\kappa_1 > x$ (by Lemma 2 b.)). Hence, the existence of an interval A^* such that $\theta(b^{A^*}) = t_n$ is ensured. This, in turn, ensures the existence of an equilibrium similar to the one described in a), which concludes the proof of step A.

Step B.: Any equilibrium b of θ satisfies $\theta(b) = m(t,\kappa)$.

For each profile (t,κ), we let $i' = \#\{i \in N \mid t_i \leq m(t,\kappa)\}$ denote the number of players with peak lower than $m(t,\kappa)$ and $j' = \#\{j \in \{1,\ldots,n-1\} \mid \kappa_j \leq m(t,\kappa)\}$ stand for the number of phantoms lower than $m(t,\kappa)$. Since (t,κ) has $2n-1$ elements, it follows that $i'+j'\geq n$ so that $n-i'\leq j'$.

Suppose, by contradiction, that there is some GAM, θ, that admits an equilibrium b with $1>\theta(b)>m(t,\kappa)$.[13] The rest of the proof inspects the different cases for each value of $n-i'$. A symmetric argument applies when $0<\theta(b)<m(t,\kappa)$.

Step B.I. $n-i'\in\{0,n\}$**.** Assume first that there is some equilibrium b with $n-i'=0$. It follows that $i'=n$ players have a peak lower than $m(t,\kappa)$. Since, by assumption, $m(t,\kappa)<\theta(b)$, Lemma 3 implies that each player i plays $b_i=[0,\theta(b)]$. However, by definition $\theta(b)$ is the α-quantile of the sample generated by b given q and η. Since $\alpha\in(0,1)$, it follows that $\theta(b)\in(0,\theta(b))$ which is impossible. If there is some equilibrium b with $n-i'=n$, then all players have a peak higher than $m(t,\kappa)$. Hence, a similar contradiction to the case with $n-i'=0$ arises, which concludes Step B.I.

Step B.II. $n-i'\notin\{0,n\}$**.** Assume now that there is some equilibrium b with $n-i'\notin\{0,n\}$ and let $i''=\#\{i\in N\mid t_i<\theta(b)\}$ denote the number of players with a peak strictly lower than the outcome $\theta(b)$. Since, by assumption, $\theta(b)>m(t,\kappa)$ it follows that $i'\leq i''$.

Given that $\kappa_j\leq\kappa_{j+1}$ for any $j\in\{1,2,\ldots,n-2\}$, $n-i'\leq j'$ and $i'\neq n$, the following inequality holds:

$$\kappa_{n-i'}\leq\kappa_{j'}\leq m(t,\kappa).$$

If $i''=n$, there are n players with a peak strictly lower than $\theta(b)$. Lemma 3 implies that each player plays $[0,\theta(b)]$, which, in turn, implies that $\theta(b)$ is in the interior of $[0,\theta(b)]$, a contradiction. Therefore, $i''\leq n-1\Leftrightarrow n-i''\geq 1$. Moreover by definition we have that $i'\leq i''\Longleftrightarrow n-i''\leq n-i'$ which implies that

$$\kappa_{n-i''}\leq\kappa_{n-i'}\leq\kappa_{j'}\leq m(t,\kappa).$$

If there is no t_h with $t_h=\theta(b)$ then by Lemma 3, i'' players play $[0,\theta(b)]$ and $n-i''$ players play $[\theta(b),1]$. Therefore, $b\in\mathscr{B}(n-i'',\theta(b))$. For b to be an equilibrium it must be the case that $\theta(b)=\kappa_{n-i''}$ which contradicts $\theta(b)>m(t,\kappa)$. Thus, there is no equilibrium b with $\theta(b)\neq t_h$.

If there is some t_h with $t_h=\theta(b)$ then i'' players play $[0,\theta(b)]$, $n-i''-1$ players play $[\theta(b),1]$ and player h plays the strategy b_h. If $b_h=[\theta(b),1]$ then $\theta(b)=\kappa_{n-i''}$ which contradicts $\theta(b)>m(t,\kappa)$. If $b_h\neq[\theta(b),1]$ then $t_h=\theta(b)<\theta([t_h,1],b_{-h})$ with $([t_h,1],b_{-h})\in\mathscr{B}(n-i'',t_h)$. Hence, by Lemma 2 b.), $t_h<\theta([t_h,1],b_{-h})\Longleftrightarrow t_h<\kappa_{n-i''}$. Thus, we have that $t_h=\theta(b)<\kappa_{n-i''}$ which contradicts $\theta(b)>m(t,\kappa)$. Thus, there is no equilibrium b with $\theta(b)=t_h$, which ends the proof. **Q.E.D.**

We now have all the tools that are necessary to state the main result of this paper.

[13]An equilibrium with $\theta(b)=1$ can be trivially ruled out since it requires that all players announce singletons. Obviously, any i with $t_i<1$ can gain by deviating to $[t_i,t_i+\varepsilon]$ for $\varepsilon>0$ and small enough.

Theorem 1. *For every generic GMR there exists a GAM that Nash-implements it.*

Proof. Take some generic *GMR* with phantom vector $p = (p_1, \ldots, p_{n-1})$. We want to prove that there is some *GAM* with phantom vector $\kappa = (\kappa_1, \ldots, \kappa_{n-1})$ that Nash-implements it. Given the result of Proposition 1, it is sufficient to show that there exists a *GAM* with $\kappa = p$.

Assume first that every $p_j \in (0,1)$. In this case, it suffices to set $\alpha \in (0,1), q \in \mathbb{R}^+$ and some function η so that, for each $j \in \{1, \ldots, n-1\}$:

$$p_j = \eta^{-1}\left(\frac{\alpha(nq+j)-(n-j)q}{(n-j)-\alpha(n-2j)}\right), \tag{1}$$

leading to $\kappa = p$ as wanted.

Assume now that there is some pair $a, b \in \{1, \ldots, n-1\}$ such that $p_a = 0$ and/or $p_b = 1$ with $p_i \in (0,1)$ if $i \in (a,b)$.[14]As previously argued, it must be the case that $p_1 \leq p_2 \leq \ldots \leq p_{n-1}$. Hence, for any $s \leq a$, $p_s = 0$ and for any $t \geq b$, $p_t = 1$.

Take now some q and α such that

$$\frac{\alpha(nq+a)-(n-a)q}{(n-a)-\alpha(n-2a)} = 0 \text{ and } \frac{\alpha(nq+b)-(n-b)q}{(n-b)-\alpha(n-2b)} = 1. \tag{2}$$

This ensures that $\kappa_a = 0$ and $\kappa_b = 1$. The previous equalities are equivalent to

$$q = \frac{a\alpha}{n(1-\alpha)-a}, \tag{3}$$

while α depends on the value of $a+b$. More precisely,

$$\alpha = 1/2 \text{ if } a+b=n, \tag{4}$$

$$\alpha = \frac{1}{(n-a-b)n}((n-a)(n-b) - \sqrt{ab(n-a)(n-b)}) \text{ if } a+b<n \tag{5}$$

and

$$\alpha = \frac{1}{(n-a-b)n}((n-a)(n-b) + \sqrt{ab(n-a)(n-b)}) \text{ if } a+b>n. \tag{6}$$

Moreover, since $0 \leq \kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_{n-1} \leq 1$, it follows that, for any $s \leq a$, $\kappa_s = 0$ and for any $t \geq b$, $\kappa_t = 1$.

If $b = a+1$, then we are done, since $\kappa = p$. If $b > a+1$, then by assumption, any p_j with $j \in \{a, \ldots, b\} \cap \{1, \ldots, n-1\}$ satisfies $p_j \in (0,1)$. Then, given that q and α are given by (1), it is enough to suitably select η such that for any $j \in \{a, \ldots, b\} \cap \{1, \ldots, n-1\}$, $p_j = \eta^{-1}(\frac{\alpha(nq+j)-(n-j)q}{(n-j)-\alpha(n-2j)})$ which ensures that $\kappa = p$ as wanted. **Q.E.D.**

Finally, we discuss some examples that show the usefulness of the analysis above. The first one is concerned with the implementation of the Condorcet winner. The second attempts to illustrate how to implement *GMRs* with interior phantoms.

Example 1: Implementing the Condorcet winner. Let $N = \{1,2,3\}$ be the set of players with $t_1 < t_2 < t_3$ and set $q = 1$, $\alpha = 1/2$ and $\eta(x) = x$. Namely, each player is endowed with a weight of $1 + \overline{b_i} - \underline{b_i}$ and the outcome selected corresponds to the median of the distribution

[14]To ensure that $\alpha \in (0,1)$, when $p_1 = 0$ and $p_{n-1} < 1$, we consider that $b = n - \frac{1}{2}$ and when $p_1 > 0$ and $p_{n-1} = 1$, we consider that $a = \frac{1}{2}$.

generated by b. For short, we let $\theta(b)$ denote the mechanism outcome and $\phi(b,z)$ the cumulative distribution associated to any profile b. The unique equilibrium outcome of this game is the selection of t_2, the median of the peaks and the Condorcet winner policy.

We first prove that t_2 is an equilibrium outcome and then show that it is the unique one. Let $b = (b_i, b_{-i})$ be a strategy profile with $\theta(b) = t_2$ for any $t_2 \in (0,1)$. If b is an equilibrium, Lemma 3 implies that $b_1 = [0,t_2]$ and $b_3 = [t_2,1]$. Thus, in order to prove that there is an equilibrium with outcome t_2, it suffices to show that there is some b_2 with $b = ([0,t_2], b_2, [t_2,1])$ satisfying $\theta(b) = t_2$. If $b_2 = [0,t_2]$, $\phi(b,t_2) = \int_0^{t_2} \frac{2+\frac{2}{t_2}}{3+2t_2+1-t_2} dx > \frac{1}{2}$ so that $\theta(b) < t_2$, whereas, if $b_2 = [t_2,1]$, $\phi(b,t_2) = \int_0^{t_2} \frac{1+\frac{1}{t_2}}{3+t_2+2(1-t_2)} dx < \frac{1}{2}$ which implies that $\theta(b) > t_2$. Therefore, player 2 can change smoothly her strategy from $[0,t_2]$ to $[t_2,1]$ and find a strategy b_2 which leads to $\phi(b,t_2) = 1/2$ –that is, to the implementation of t_2. The precise strategy of player 2 depends on the value of t_2. For each $t \in (0,1)$, let $w(t) = \sqrt{1-2t+4t^2} + 2t - 1$. If $t_2 \leq 1/2$, then $b_2 = [0, w(t_2)]$ ensures that $\theta(b) = t_2$, whereas, when $t_2 \geq 1/2$, then $b_2 = [1 - w(1-t_2), 1]$ ensures that t_2 is the outcome.

Now, in order to prove that there is no other possible equilibrium outcome, assume by contradiction that there is some $z \neq t_2$ elected at some equilibrium b. Assume that $z < t_2$, the case with $z > t_2$ being symmetric. If $z \in (t_1, t_2)$, then, due to Lemma 3, $b = ([0,z],[z,1],[z,1])$ since $t_1 < z < t_2 < t_3$. But then $\phi(b,z) = \int_0^z \frac{1+\frac{1}{z}}{5-z} dx$ so that $\phi(b,z) < \frac{1}{2}$ for any $z \in (t_1,t_2)$. In other words, it is not possible that such a strategy profile is an equilibrium. If $z \in [0,t_1)$, then b must be such that the three players play $[z,1]$ which leads to an outcome larger than z. Finally, if $z = t_1$, then in any equilibrium b, players 2 and 3 play $b_2 = b_3 = [t_1,1]$. However, if player 1 plays $[0,t_1]$, then $\phi(([0,t_1],b_2,b_3),t_1) < \frac{1}{2}$ for any $t_1 \in [0,1]$. Thus, $\theta([0,t_1],b_2,b_3) > t_1$. Yet, since $[0,t_1]$ is the best response of any player with peak to the left of t_1, it follows that $[0,t_1] \in \arg\min_{b_1 \in \mathscr{B}} \theta(b_1,b_2,b_3)$. Therefore, $\theta(b_1,b_2,b_3) > t_1$ for any $b_1 \in \mathscr{B}$ so that there is no equilibrium with outcome t_1. All in all, the unique equilibrium outcome associated with θ is t_2.

To see how the previous argument extends to any number of players, consider that the number of players is odd.[15] We let $n = 2k+1$ for some non-negative integer k and let t^* denote the median peak of the n players. It follows that there are exactly k players with a peak smaller than t^* and k players with a peak larger than t^*. We now set $q = k$, $\alpha = 1/2$ and $\eta(x) = x$. In other words, each player is now endowed with a weight of $k + \overline{b_i} - \underline{b_i}$ and the outcome selected corresponds to the median of the distribution generated by b. Using the equalities (3) and (4) in the proof of Theorem 1, the previous specifications ensure that half of the phantoms are located at zero and half of them at one, which leads to the implementation of the median. In equilibrium, the k players with a peak smaller than t^* approve of $[0,t^*]$ whereas the k ones with a peak larger than t^* approve of $[t^*,1]$. The player

[15]This ensures the existence of a unique median player (or Condorcet Winner). When $n = 2k$ for some $k > 1$, there are two median players: player k and player $k+1$. Setting $a = k$ and $b = k+1$ and replacing these values in equalities (3) and (6) leads to the *GAM* with parameters $q = \frac{\sqrt{k^2-1}}{2} + \frac{k-1}{2}$ and $\alpha = \frac{q}{2q+1}$ and $\eta(x) = x$ for any $x \in [0,1]$. This *GAM* implements the peak of the leftist median. Conversely, setting $a = k-1$ and $b = k$ in the equalities (3) and (5) leads to the *GAM* with parameters $q = \frac{\sqrt{k^2-1}}{2} + \frac{k-1}{2}$ and $\alpha = \frac{kq+q}{2kq+k-1}$ and $\eta(x) = x$ for any $x \in [0,1]$. This *GAM* implements the peak of the rightist median player.

with peak at t^* just needs to play some strategy b_i^* that ensures that the median of the profile equals t^*. As in the case with just three players, one can prove by continuity that such strategy exists since the median of the score distribution is smaller than t^* when he plays $[0, t^*]$ and larger than t^* when he plays $[t^*, 1]$.

Example 2: A GAM with interior phantoms. If we set $q = 0$, $\alpha = \frac{1}{2}$ and $\eta(x) = x$, we get the Median Approval mechanism discussed in Section 3. The phantoms of this Approval mechanism must satisfy, for any $j \in \{1, 2, \ldots, n-1\}$, $\eta(\kappa_j) = \frac{j+q(2j-n)}{n} \iff \kappa_j = \frac{j}{n}$ (as already shown in Section 3, when $N = \{1,2,3\}$ we have $\kappa_1 = \frac{1}{3}$ and $\kappa_2 = \frac{2}{3}$). The equilibria with this mechanism in the case in which $m(t_1, t_2, t_3, \frac{1}{3}, \frac{2}{3})$ equals one of the peaks is similar to the ones of the Approval mechanism that implements the Condorcet winner.

However, in the precise case in which $m(t_1, t_2, t_3, \frac{1}{3}, \frac{2}{3}) = \frac{1}{3}$ (the case $m(t_1, t_2, t_3, \frac{1}{3}, \frac{2}{3}) = \frac{2}{3}$ being symmetric), the logic is different. Indeed, the mechanism admits a unique equilibrium b^* with $b_1^* = b_2^* = [0, \frac{1}{3}]$ and $b_3^* = [\frac{1}{3}, 1]$. In general, if the equilibrium outcome coincides with a phantom and not with a type, there is a unique equilibrium (all players playing either to the left or to the right of the outcome) whereas this is not the case when a player's peak is the equilibrium outcome (this player can play in several ways, while the rest of the players play either to the left or to the right of the outcome).

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Figures

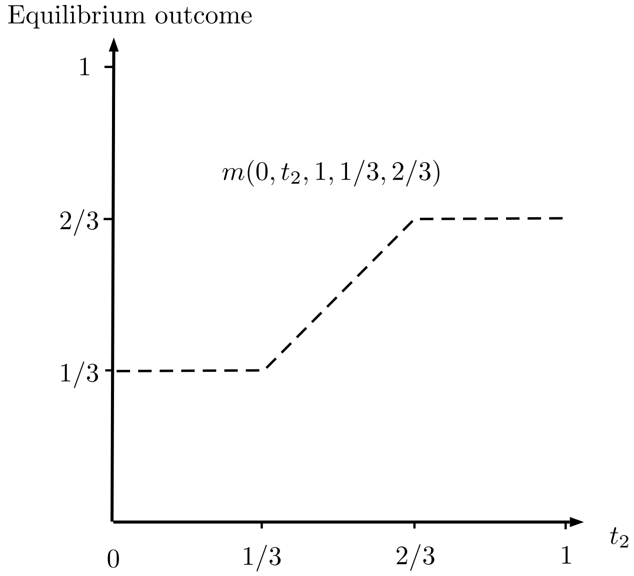

Figure 1. Equilibrium outcome as a function of t_2.

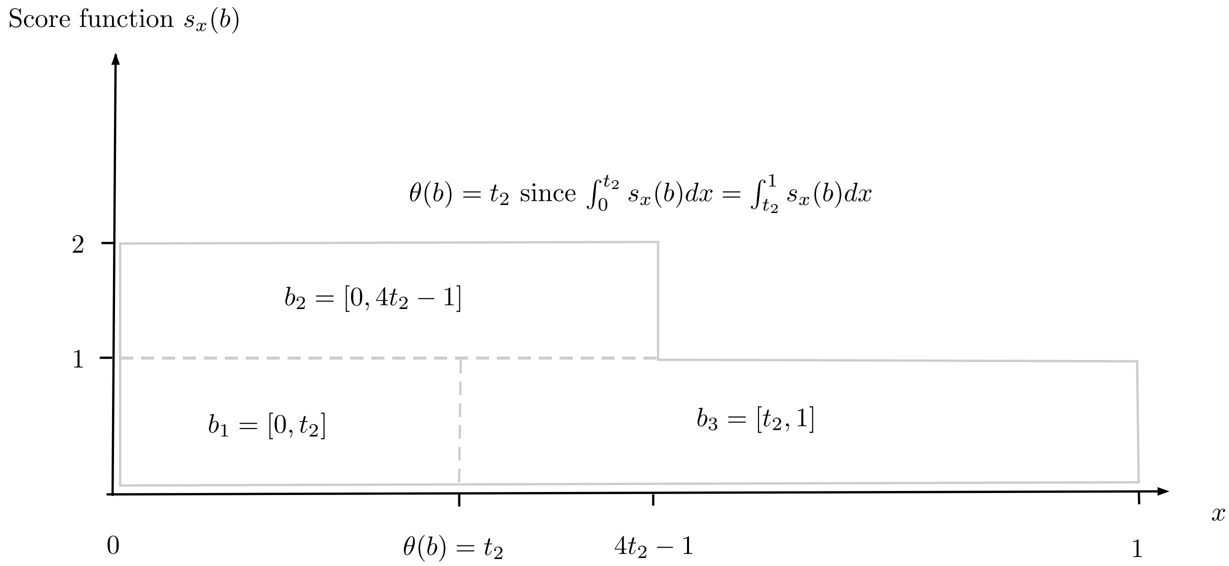

Figure 2. An equilibrium strategy profile that implements the ideal policy of the median voter.