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RESEARCH

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Abstract

We study the implications of delegating bids to a *bidding agency* for the revenues and efficiency of the *Generalised Second-Price* auction, the standard sales mechanism for allocating online ad space. The *agency* maximises both its own profits and the advertisers' surplus and implements collusive agreements by means of side contracts. Despite the specificity of the auction mechanism, we show that an *agency* can profitably deliver bid delegation services, which increase the advertisers' surplus and contribute to market efficiency if ad spaces are not too different. The optimal policy of the *agency* coordinates advertisers on a unique efficient equilibrium, which implies a lower bound on the search engine's revenue and makes the lowest-valuing member of the coalition indifferent between refraining from participating the auction and defecting the agreement. We also point out that the *Vickrey-Clarke-Groves* is a compatible solution and is uniquely achieved by bid delegation if side contracts are based on the *locally envy-free* stability criterion.

Keywords: Generalised second-price auction; Position auction; Bidding ring; Cartel; Bidding agency; Sponsored Search

JEL Classification: D44, C72, M3, L41

1 Introduction

Billions of simultaneous sponsored search auctions — called *Generalised Second-Price* (*GSP*) auction — are held daily to allocate bundles of online ad spaces by Google, Microsoft and Yahoo![1,2]. Each time a customer makes a search query, an automated auction is triggered putting advertisers in competition for ad space. This sales mechanism has become the main source of revenue for search engines and is the most widely used auction ever designed. In 2016, earnings from advertising represented more than 90% of Google's revenues in an online advertising market that in the same year was worth nearly $190 billion[3].

The unique quality of ad space is that its value (the expected return on investment) is determined by consumers search behaviour, which constitutes complex and costly information for firms without the skills to assess. Collecting information on consumers online behaviour has became a key driver of competition between firms and the novel strategic considerations it represents has led to a concentration and an outsourcing of marketing activities to third-parties. An increasing number of advertisers delegate their search marketing and paid search strategies to specialised marketing agencies — we call *bidding agencies* — which themselves belong to ad networks in which such activities are conducted. For instance, Gartner's U.S. Digital Marketing Spending report for 2013 documents that from a pool of 243 firms more than 50% delegate their search marketing activities[4]. These agencies offer skills and services to internalize advertisers' search costs by managing in their name their bidding strategies and advertising campaigns. This means a *bidding agency* is likely to end up acting on behalf of different advertisers during the same set of keyword auctions, and so it can affect both allocative and revenue performance of the sales mechanism by coordinating advertisers' strategies.

Bid delegation helps reducing the knowledge gap between advertisers and the search engine, which should enhance the level of competition and bidders participation in each instance of auction. Meanwhile, it possibly undercuts search engines ability to extract surplus from advertisers, which raise questions regarding the way it is shared out between advertisers, the search engine and the *agency*. The main backdrop is that by bidding on behalf of different advertisers during the same keyword auctions, the *agency* is at liberty to manipulate its clients final payments by coordinating individual bids, which in the spirit of classic collusion games relies on distorting the rivalry between advertisers. Thus, in a market essentially driven by the Google-Facebook duopoly, and which generates around 20% of the global ad spending in 2016, it is clearly important to understand how this trend in delegation affects market organisation, efficiency and the way rent is allocated among players.

In this paper, we undertake a theoretical exploration of the way in which advertiser collusion — through bid delegation — jeopardises the revenues and distorts rent allocation of the *GSP* auction by considering a *bidding agency* that seeks to minimise its clients' rivalry and final payments but also to

[1] The auction works as follows: advertisers announce the maximum price they are willing to pay for each click on their ad, referred to as their bid for a click, and they pay a price (per click) equal to the bid announced by the advertiser assigned to the position just below them. In this way, the highest bidder is allocated the ad slot at the top of the page receiving the highest number of clicks, the second-highest bidder is allocated to the second position receiving the second-highest number of clicks, and so on.

[2] A recent exception to this is provided by Facebook, which adopted the *Vickrey-Clarke-Groves* (*VCG*) mechanism in 2015.

[3] See the eMarketer's Worldwide Ad Spending report and the Google annual report 2016.

[4] Another survey, run in 2013 by Constant Contact, found that in a pool of 1305 small firms nearly 35% outsourced their bid management activities.

maximise its own profits. In order to do so, we introduce a collusive device, grounded by side payments between advertisers, to the model developed originally by Varian (2007) and Edelman et al. (2007) and we limit our attention to a simultaneous two-position game with three advertisers under the same assumption of complete information[5]. Retaining the complete information setting allows us to eliminate the issue of private information extraction encountered in the literature on bidding rings as the *bidding agency* has full knowledge of each member's private valuations[6]. Moreover, such an assumption is motivated by the fact that, when outsourcing their marketing activities, advertisers commit themselves in full information disclosure to the *bidding agency*.

We focus on a collusion enhanced by a *pre-communication* phase and the market organisation we consider has several features in common with menu auction and common agency games, as analysed by Bernheim and Whinston (1985, 1986a,b), in which several principals (bidders) delegate their right to make a decision to a single common agent. The *bidding agency* is assumed to be risk-neutral, to support no unit cost and to be residual claimant of the gain from delegation. However, in contrast with the standard common agency approach, in which bidders offer non-cooperative schedules to the agent, the *bidding agency* makes a contract proposal to which it commits itself. In the model, the bidding contract includes a bid profile to be placed on the client's behalf, a management fee and the way in which side payments are to be implemented[7]. The contract embodies an *allocation rule* which ensures a monetary payback to each member and is computed according to each contribution to the collusion. Contributions are determined as the difference in the prices from which the members benefit when the collusion is in operation. These are assumed to be defined *prior* to the auction and paid upfront by each member. Then, the *bidding agency* uniformly divides the surplus generated by the delegation and extracts a share of this surplus by imposing a fix fee.

The specificity of the *GSP* mechanism makes it difficult to study agency games in this setup. In contrast with standard mechanisms, the rules of the auction do not permit one bid per position, all advertisers seek the top position and incentives to defect from a weak agreement are high. This means that standard collusive strategies encountered in multi-unit second-price auctions do not hold in this context. Indeed, in a multi-unit Vickrey auction, bidders offer multiple bids, one for each unit. It is then easy to reach a collusive outcome in which all units are splitted among bidders and the auction ends-up on an inefficient equilibrium (see Milgrom (2000)). Moreover, the *GSP* auction does not inherit the strategy-proofness property of the Vickrey mechanism and it is well-known that this complex environment generates a multiplicity of — asymmetric and symmetric — non-cooperative equilibria (see Börgers et al. (2013)). In the present model, collusive bids depend, by way of individual contributions and members' outside options, on non-cooperative outcomes so that the collusive game also supports multiple equilibria. Furthermore, even though the *bidding agency* has full knowledge of its clients' private valuations, the

[5]We consider an all-inclusive collusion, in which reducing to the smallest set of advertisers and positions does not affect the soundness of the model and where the main insights carried can be applied to more positions and players.

[6]See, for instance, Graham and Marshall (1987), McAfee and McMillan (1992), Mailath and Zemsky (1991), Marshall and Marx (2007) and Brusco and Lopomo (2002).

[7]Various strategies can be adopted. One might include a bid rotation mechanism so that the agency assigns different advertisers to different keywords and rotates these assignments at each auction. Another might use "split award" agreements and hold as fixed the set of auctioned keywords for which advertisers compete, or not so that each keeps its own bundle of keywords. We find the strategy of maintaining advertiser competition for the same set of keywords to be technically the easiest way of achieving efficient coordination and the most suitable for the specific multi-object environment considered here.

virtual environment makes it difficult, if not impossible, to control for fake aliases and possible shill bids from them in order to pocket a position they couldn't have won non-cooperatively. The proposed bidding contract is thus susceptible to *ex-post* defections (i.e., once a contract has been settled) by advertisers. In this context, incentive compatibility relies on finding a suitable way to prevent bidders to double-cross the *agency*'s actions during the main auction, and the *agency* has to implement a compensation policy that keeps each player's interests aligned under this constraint.

The present model suggests that the *agency* can profitably deliver bid delegation services. Its presence in the market increases the advertisers' surplus and contributes to market efficiency. Bid delegation implements an efficient collusion constrained by the *agency*'s participation fee, which limits the adverse effect of a complete cartel on auction revenues and rent expropriation.

Our first finding is that bidders can efficiently collude by means of bid delegation provided that substitutability between positions is sufficiently high. If this condition is met, we show that the *agency* coordinates advertisers on an efficient outcome which entails higher surplus than the smallest Nash equilibrium in price. Precisely, among the multiple equilibria, bid delegation pins down the outcome to a low-revenue equilibrium in which all the rent is captured by advertisers. If the condition is not met and positions differ too much in terms of rentability, even if the *agency* is able to coordinate advertisers an a better equilibrium, we find that the resulting outcome is inefficient as positions are not allocated in decreasing order of advertisers valuations. Hence, bid delegation solves the multiplicity problem and ensures the efficiency of the *GSP* mechanism if positions are not too differentiated.

Second, we provide a characterisation of the equilibrium bids to be played and identify a trade-off faced by the *agency* when setting the management fee: on the one hand, since equilibrium bids are non-decreasing functions of the commission fee, a high fee allows the extraction of more surplus from advertisers but reduces the gain from collusion as internal rivalry increases; on the other hand, a sufficiently low fee increases the surplus available to advertisers as internal rivalry decreases but reduces, *de facto*, its ability to extract rent. The optimal policy for the *agency* is then to set a fee that makes the lowest-valuing member of the coalition indifferent between staying out the auction or breaking the collusion. This critical threshold represents a limit case at which internal competition can be annihilated by the *bidding agency*, which constrains its behaviour and the outcome of the collusion.

The *bidding agency* and the search engine hold congruent interests. The *bidding agency*'s profits are non-monotonic in the fee, and it is able to set an optimal level of fee that secures a minimum revenue for the seller without weakening incentives to collude. Thus, the search engine's revenue is increasing in the participation fee and it benefits from the activity of the *bidding agency*. However, an increase in the fee drives up collusive prices. At the limit case, collusive prices converge to the one that sustain the smallest, possibly inefficient, non-cooperative equilibrium. As a result, a bid delegation organised by an absolute rent-seeking *agency* implies an upper bound to the seller's revenue. From a regulation perspective, in line with Bulow and Klemperer (1996), it can be argued that in the short run, incentives for a seller to break collusion are mitigated if its focus is on capturing market shares in that it might be achieved at the expense of allocative efficiency, the advertisers' surplus and participation level.

We last discuss whether the *Vickrey-Clarke-Groves* (*VCG*) solution can be compatible and implemented by a rent-seeking *bidding agency*. A well-known result forwarded by the literature is that the *GSP* auction has a full-information *locally envy-free* (*LEF*) equilibrium that is *VCG*-equivalent in price and allocation. The *LEF* equilibrium refinement, on which the equivalence relies, states that bidders should choose the position that maximises their profits taking prices as given. In equilibrium, no bidder should find it profitable to swap his position with that of a bidder allocated just above them. We import this best-response flavor to our setup by considering that the *agency* picks the prices that maximise bidders' individual profits in their names. When contemplating some different positions (and so cheating on the agreements), taking each position price as given, incentives compatibility constraints are then consistent with the *LEF* refinement. We find that advertisers do not have sufficient incentives to reduce their expressed demand, so that the *GSP* auction becomes collusion-proof. Independently of the level of the participation fee, the *VCG* solution is uniquely determined by the process of coordination.

Related works Despite its importance and potentially adverse outcomes, advertising intermediation, and bid delegation, has been overlooked by the main strand of research analysing sponsored search auctions, as pioneered by Aggarwal et al. (2006), Varian (2007) and Edelman et al. (2007). Leaving apart the increasing role third-parties and ad networks play in this market, studies in the literature have examined such questions as the optimal design of the auction mechanism (e.g., Edelman and Schwartz (2010)), the introduction of consumer search (e.g., Athey and Ellison (2011)), search engine competition (e.g., Ashlagi et al. (2011)), budget constraints (e.g., Ashlagi et al. (2010)) and more recently, incomplete information settings (e.g., Gomes and Sweeney (2014)).

To the best of our knowledge, only two studies can be directly related to our work. Both examine the decentralisation of advertisers' bidding strategies to a third party in sponsored search auctions, but in contrast to the present work they consider either different goals or different settings. Ashlagi et al. (2009) were the first to study a position auction game in which an incentiveless third-party manages to coordinate the bidding strategies of a pool of advertisers by means of the solution concept of a *mediated equilibrium* as first proposed by Monderer and Tennenholtz (2009). The focus of Ashlagi et al. (2009) was then to study the ability of the third-party to implement the *VCG* outcome with incomplete information. Decarolis et al. (2017) also consider bid delegation to a marketing *agency* that bids on behalf of different advertisers in the same keyword auction. In their model, the *agency* is incentiveless and acts as a social planner whose main goal is to maximise the advertisers' surplus.

We depart from the previous works in a number of ways. We consider an additional aspect in the formation of agreements between advertisers and the *agency*. In our model the *agency* provides a monetary payback to advertisers so as to align incentives for reaching an agreement and seeks to maximise both the advertisers' surplus and its own profits. In contrast with the aforementioned works, we consider that the *agency* extracts a management fee for intermediation on the collusive gains in order to maximise its own revenue, which is in line with common practice in this industry[8,9]. We are thus able to account for the effects of the *agency*'s behaviour on incentives to sustain the collusive agreement and to give an analytic representation of how the seller's revenue reacts.

Next, third-parties operate along with outside advertisers, coordinated and non-cooperative bids in-

[8]An advertiser can be charged each time there is a click on the ad being managed; or the *agency* can base a mark-up on the entire amount spent (budget) on the ad; or it can also require a mark-up toward the profits that the advertiser makes on each purchase. Another interesting but more complex system, is to *pay for performance*. Here, a metric is defined *ex-ante* and the advertiser only pays once a sale is made or by employing another metric resulting from the management process.

[9]The 16th Edition of the Association of National Advertisers' (ANA) report *Trends in Agency Compensation Survey* (2013), documents that over 80% of 98 respondents claimed to use a fixed-fee compensation policy.

teract, which may lead to tractability issues. Decarolis et al. (2017) manage to integrate this interaction by restricting the behaviour of outside bidders to that of the *locally envy-free* refinement of Varian (2007) and Edelman et al. (2007). To avoid these difficulties, we consider that a monopolistic *agency* is present on the market and suppose all-inclusive collusion[10]. Such an environment is optimal as the *agency* only has to control its members' bids and does not have to anticipate the bids of potential outside advertisers. We implicitly assume here that if a profitable, efficient collusion cannot be implemented under the all-inclusive assumption, we can reasonably expect it not to be the case if we allow for incomplete cartels.

Finally, there is a recent strand of literature, including studies by Feldman et al. (2010) and Balseiro and Candogan (2017), that focuses on bid delegation to *agencies* in the context of multi-sided ad-exchange platforms, the second main segment of the advertising market. In this market, advertisers seek to attract the attention of internet users by placing advertisements in an ad space provided by a publisher on a webpage. Ad-exchanges bring publishers and advertisers together in the same marketplace in which an ad space is sold using a single-unit Vickrey auction[11]. In practice, advertisers belong to a network handled by an *agency* that buys ad space on their behalf directly on the exchange platform. In contrast to our model, in addition to the fact that we consider a multi-unit auction, these studies analyse delegation to an *agency* that has to select only one bid from its portfolio of advertisers and does not make side payments.

2 A position auction with a ***bidding agency***

We model the *GSP* auction as a game involving an all-inclusive cartel (or *ring*) of three bidders who compete for two positions $k \in \mathcal{K} = \{1, 2\}$ sold simultaneously by a search engine. A third-party, taking the role of what we call the *bidding agency*, coordinates the collusion. Each position k has an associated, commonly known, expected *click-through rate* (henceforth *ctr*) denoted by $\alpha_k \geq 0$ with $\alpha_1 > \alpha_2 > \alpha_3 = 0$.

Advertisers A player's valuation expresses his willingness to pay for a click and is denoted by x_i, with $x_i \in [0, \bar{x}]$. Let $\boldsymbol{x} = \{x_1, x_2, x_3\}$ be the set of individual valuations, labelled in decreasing order so that $x_1 > x_2 > x_3$. Valuations are understood to be independent of the positions, the identities of the advertisers allocated and of the customers' clicking behaviour[12]. Henceforth, i's value of being allocated in position 1 or 2 is defined by the product $\alpha_k x_i$. Each player simultaneously submits a single-dimensional and non-negative bid $b_i \geq 0$ for a click as a function of his valuation. To distinguish between a *non-cooperative* (or *Nash*) bid and a *collusive* bid we denote the former by b_i and the latter by $b_i^{\mathcal{N}}$. Then, let $\boldsymbol{b} = \{b_1, b_2, b_3\}$ be the set of bids and $\boldsymbol{b}_{-i}$ the set of bids excluding b_i and let $\boldsymbol{b}^{\mathcal{N}} = \left\{b_1^{\mathcal{N}}, b_2^{\mathcal{N}}, b_3^{\mathcal{N}}\right\}$ be the set of possible actions for cartel members, with $\boldsymbol{b}_{-i}^{\mathcal{N}}$ defined in a similar fashion. The game consists of allocating positions to bidders based on the order of their bids (or *expressed demand*). According to the auction's rules, when bidder i is assigned a position k he is charged a price *per* click $p_k = b_{k+1}$. Thus,

[10]This assumption is motivated by empirical observations that during the most important keyword auctions, advertisers are mostly represented by just one agency, see Decarolis et al. (2017) and Decarolis and Rovigatti (2017).

[11]See McAfee (2011) and Muthukrishnan (2009) for an approach to the practical design of this market.

[12]We do not consider the question of the allocative externality generated, for instance, by a firm's reputation among its customers. A firms of high repute might imply more clicks for a firm of low repute if the former is placed at a position just above the latter. More experienced customers appear to click more carefully and also more frequently on ads placed in the median position. The quality of the ad and a firm's reputation affect their choices. Then valuations for clicks could be allowed to vary non-linearly between positions.

the net payoff of player i when assigned position k is equal to,

$$\pi_i = \alpha_k \left(x_i - b_{k+1}\right)$$

Consider that bidders meet before the main auction starts. If they attempt to outbid each other during this auction they will give most of the surplus to the seller. The task then is to reach an agreement whereby they can limit surplus extraction and expropriate a significant share of that surplus from the seller. The goal of the *bidding agency* is to coordinate matters in such a way that no bidder finds it profitable to break the settlement by reverting to competitive behaviour. The objective, therefore, is to draw up individual contracts so that each potential member is better off than they would be in the absence of coordination. The contract comprises a bid recommendation, the side payments that are made and the flat-fee imposed by the *agency*.

Side payments We assume that the sustainability of the cartel (or *ring*) is based on the presence of side payments. This takes the form of a monetary transfer ω_k, corresponding to a bidder's contribution, from each member to the *bidding agency*. In return, the *bidding agency* makes a lump-sum transfer τ to each member comprising the collusive benefits[13].

Remark 1. *Individual contributions are based on the final allocation that is anticipated and incurred as a* sunk cost *by members.*

Contributions ω_k are computed as the difference between the price bidder i would have to pay to the seller in the absence of a *cartel* and the price he actually pays in the presence of the *cartel*. Let $\hat{p}_k$ be the payment made by member i to the seller when assigned position k in the presence of collusion. Then,

$$\omega_k(.) \quad = \quad \max\left\{\alpha_k\left(p_k - \hat{p}_k\right); 0\right\} = \max\left\{\alpha_k\left(b_{k+1} - b_{k+1}^{\mathcal{N}}\right); 0\right\}$$

The expected total gain $\Pi_{\mathcal{N}}$ from colluding (referred to as *spoils*), and which is to be *uniformly* redistributed at the end of the auction, is the sum of each member's payment to the *bidding agency*,

$$\Pi_{\mathcal{N}} = \sum_{k \in \mathcal{K}} \omega_k(.)$$

At the end of the auction, the *bidding agency* exerts a credible monopoly power by setting a fee $\varepsilon \in [0,1]$ upon $\Pi_{\mathcal{N}}$. It may (i) act as a risk-neutral, incentiveless agent when $\varepsilon = 0$ or (ii) levy a fee $0 < \varepsilon < 1$ upon collusive profits, or (iii) keep the entire collusive gain with $\varepsilon = 1$ and so substitute the seller. The quantity ε is set *non-strategically* and each member receives a transfer τ equals to:

$$\tau = \frac{(1-\varepsilon)}{3} \sum_{k \in \mathcal{K}} \omega_k(.) = \frac{1-\varepsilon}{3} \sum_{k \in \mathcal{K}} \alpha_k \left(b_{k+1} - b_{k+1}^{\mathcal{N}}\right) \tag{1}$$

We call this quantity a *uniform-ε redistribution rule*. It is assumed as being retained by the *bidding*

[13] This is done at the end of the main auction so that the *bidding agency* holds the bargaining power and necessarily controls the incentive compatibility constraints.

agency if a defection occurs. Given the redistribution scheme, the *agency*'s profits are given by,

$$\Gamma_{\mathcal{N}} = \sum_{k\in\mathcal{K}} \omega_k(.) - \sum_{i=1}^{3} \frac{(1-\varepsilon)}{3} \sum_{k\in\mathcal{K}} \omega_k(.) = \varepsilon \sum_{k\in\mathcal{K}} \omega_k(.)$$

The total payment $p_i^{\mathcal{N}}$ made by player i to the cartel:

$$p_i^{\mathcal{N}} = \begin{cases} \omega_k(.) - \tau & \text{if } k = \{1,2\} \\ -\tau & \text{if } k = \{\emptyset\} \end{cases} \tag{2}$$

that is, bidder i if assigned to position $k = \{1,2\}$ pays his individual contribution to the cartel and receives his individual share from the *bidding agency* in return, whereas if $k = \{\emptyset\}$, he receives his individual share but does not contribute[14]. As a result, member i's expected gain assigned to position k equals:

$$\pi_i^{\mathcal{N}} = \begin{cases} \alpha_k \left(x_i - b_{k+1}^{\mathcal{N}}\right) - p_i^{\mathcal{N}} & \text{if } k = \{1,2\} \\ -\tau & \text{if } k = \{\emptyset\} \end{cases}$$

Agency problem The *agency* offers individual contracts $\gamma_i = \left(\varepsilon, \left\{b_i^{\mathcal{N}}, p_i^{\mathcal{N}}\right\}\right)$, to which it is committed, composed of a system of recommended non-negative bids $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_i^{\mathcal{N}}\right)_{i\in\mathcal{N}}$ to be placed during the auction, the share $\varepsilon \in [0,1]$ it intends to keep and the side-payments required from each member $\boldsymbol{p}^{\mathcal{N}} = \left(p_i^{\mathcal{N}}\right)_{i\in\mathcal{N}}$. Figure 1 illustrates the market structure we consider. To illustrate the idea, the timing of the game can be described as follows:

1. Before the auction, the *agency* makes a contract proposal $\gamma_i = \left(\varepsilon, \left\{b_i^{\mathcal{N}}, p_i^{\mathcal{N}}\right\}\right)$.

2. In this *participation phase*, each member either accepts or rejects the proposal. If a member rejects the proposal then no delegation takes place and each member plays non-cooperatively; otherwise, bid delegation comes in to operation.

3. Members are then asked to pay their contributions ω_i and to refrain from bidding in their own name at the main auction. At this *deviation phase*, they can defect from the agreement and bid some $\tilde{b}_i$ in order to win a position they could not have won if all were to play non-cooperatively.

4. The seller then allocates the bidders in decreasing order of their bids and each is charged $\hat{p}_k$.

5. If no defection occurs during the auction stage, the *bidding agency* redistributes τ and nothing else.

The *agency* maximises the advertisers' total surplus, subject to incentive compatibility,

$$\pi_i^{\mathcal{N}}\left(b_i^{\mathcal{N}}, \boldsymbol{b}_{-i}^{\mathcal{N}}, x_i\right) - \tilde{\pi}_i\left(\tilde{b}_i, \boldsymbol{b}_{-i}^{\mathcal{N}}, x_i\right) \quad \geq \quad 0, \quad \forall i = \{1,2,3\}, \forall b_i^{\mathcal{N}} \in \boldsymbol{b}^{\mathcal{N}} \tag{3}$$

[14]The environment of complete information simplifies greatly the functioning of such a redistribution rule as there is no need to implement a *pre-knockout* in order to make players reveal their private willingness to pay for a click. There is no adverse selection in this framework and no issue of cartel misrepresentation at the main auction.

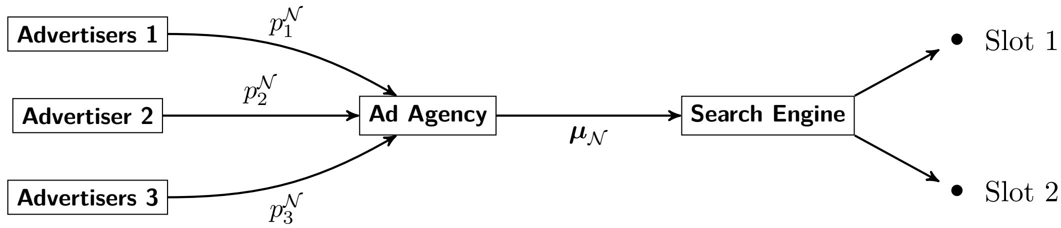


Figure 1: Market structure

in which $\tilde{\pi}_i\left(\tilde{b}_i, \boldsymbol{b}_{-i}^{\mathcal{N}}, x_i\right)$ represents player i's surplus if he exploits the collusion with a bid $\tilde{b}_i \neq b_i^{\mathcal{N}}$ while the others play according to the strategy $b^{\mathcal{N}}$. Profits from a deviations to a lower or an upper position take the following formulation:

$$\begin{aligned} \tilde{\pi}_i\left(\tilde{b}_i, .\right) &= \alpha_l\left(x_i - \max\left\{b_{l+1}^{\mathcal{N}}, 0\right\}\right) - \omega_k(.), \quad \forall l > k && (4) \\ \tilde{\pi}_i\left(\tilde{b}_i, .\right) &= \alpha_s\left(x_i - \max\left\{b_s^{\mathcal{N}}, 0\right\}\right) - \omega_k(.), \quad \forall s < k && (5) \end{aligned}$$

and subject to individual rationality:

$$\pi_i^{\mathcal{N}}\left(b_i^{\mathcal{N}}, \boldsymbol{b}_{-i}^{\mathcal{N}}, x_i\right) \geq \pi_i\left(b_i, \boldsymbol{b}_{-i}, x_i\right) \tag{6}$$

and the restriction that $b_1^{\mathcal{N}} \geq b_2^{\mathcal{N}} \geq b_3^{\mathcal{N}}$.

The incentive compatibility constraint (3), (4) and (5) expresses the idea that once bidder i receives his recommended bid (prior to playing in the auction), he has the choice of either obeying or adjusting his bid against the bids the *bidding agency* places on behalf of his competitors[15]. If a deviation occurs, the defector does not receive his individual compensation τ but still incurs his contribution as a *sunk cost*. This constraint requires that there is no deviating bid $\tilde{b}$ that allocates bidder i any different position so that he is better off. The last constraint (6) means that player i has to find it profitable to join the cartel. His profits from accepting contract γ_i are at least as high as his outside option, which corresponds to his non-cooperative profits so that each outside option is assumed to be an equilibrium outcome.

Equilibrium criterion

Definition 1. *A collusive mechanism* $\boldsymbol{\zeta} = \left(\boldsymbol{\mu}_{\mathcal{N}}, \boldsymbol{p}^{\mathcal{N}}\right)$ *is an equilibrium profile if it is* (i) *individually rational and* (ii) *incentive compatible.*

In single unit auctions, a collusive device is said to implement an efficient outcome if the highest-valuing member of the collusion represents the cartel, wins the object and the cartel is able to suppress internal competition. Such an outcome naturally maximises social welfare as the good is allocated to the highest-valuing player. We modify this definition slightly to make it coherent with the *GSP* auction context.

Definition 2 (*First-best outcome*)**.** *A collusion in the* GSP *auction is an efficient mechanism if:* (i) *the final allocation is such that only the two-highest valuing members are active during the targeting*

[15] Assuming they behave according to the recommendation.

auction,(ii) it suppresses competition from bidders not allocated to any position, (iii) it maximises social welfare.

2.1 Benchmark — Non-cooperative outcome

We first give the set of *Nash* equilibria in which bidders do not collude. Renumber players so that x_k is the valuation of the bidder assigned to position k. An equilibrium outcome will be supported by some bid profile $\boldsymbol{b} = (b_1, b_2, b_3)$ if under it (i) no player can profitably deviate from the position to which he is assigned in equilibrium to any lower or higher positions,

$$\begin{aligned} \alpha_k \left(x_k - p_k\right) &\geq \alpha_l \left(x_k - p_{l-1}\right), \quad \forall l < k \\ \alpha_k \left(x_k - p_k\right) &\geq \alpha_s \left(x_k - p_{s+1}\right), \quad \forall s > k \end{aligned}$$

and (ii) an assigned player cannot profitably deviate to win any position; that is,

$$\alpha_k \left(x_k - p_k\right) \geq 0$$

with $p_k = b_{k+1}$[16].

Uncoordinated bids: The following strategy profile $\boldsymbol{b} = (b_1, b_2, b_3)$ characterises all the *Nash* equilibria of the static *GSP* auction with complete information:

$$\begin{aligned} b_1 &\in \left[\max\left\{x_2 - \frac{\alpha_2}{\alpha_1}\left(x_2 - b_3\right), x_3\right\} \,;\, \bar{x}\right] \\ b_2 &\in \left[\max\left\{x_3 \,,\, b_3\right\} \,;\, x_1 - \frac{\alpha_2}{\alpha_1}\left(x_1 - b_3\right)\right] \\ b_3 &\in \left[\max\left\{0 \,,\, x_1 - \frac{\alpha_1}{\alpha_2}\left(x_1 - b_2\right)\right\} \,;\, \min\left\{x_1 \,,\, x_2\right\}\right] \end{aligned} \tag{7}$$

Each player can envision a multiplicity of best-responses to each other's equilibrium strategy. Equilibrium bids are bounded above and below by a combination of the lowest bidder's bid and the valuations of the player above and below him. Two things are worth noticing about bid profile $\boldsymbol{b}$. First, it does not rule out inefficient and non-assortative allocations. In fact, the only allocation that is ruled out is that in which the highest-valuing advertiser wins no position and the lowest-valuing advertisers wins the top position[17]. Secondly, overbidding could be a candidate for a *Nash* equilibrium so that it does not restrict attention to undominated strategies.

[16]Note that the shape of prices is modified as upward deviation occurs, which is not in the spirit of usual competitive equilibrium analysis. Players can influence the price they pay at the end and these prices are not taken as given. Each bidder when contemplating an upper position does not expect to pay the same price as that already assigned to that position.

[17]Indeed, if the highest-valuing player was to win no position then it should be the case that $b_3 \geq x_1 \geq x_3$ which cannot be an equilibrium profile. If the lowest-valuing player was to now win the top position then it should be the case that $p_3 = b_1 > x_3$, implying a strict loss for him.

VCG outcome: According to Varian (2007) and Edelman et al. (2007), a strategy profile $\boldsymbol{b} = (b_1, b_2, b_3)$ is a *locally envy-free* or a *symmetric* equilibrium if it holds that

$$\alpha_k (x_k - p_k) \geq \alpha_{k-1} (x_k - p_{k-1}), \quad k = 1, 2$$

with $p_k = b_{k+1}$.

This criterion implies a shift in the *IC* conditions so that upward and downward deviations become symmetric. Even if advertisers were to swap each other's position, the price structure would not change. If the bidder in position k decides to undercut the bidder assigned to a position $l < k$, he can expect to pay $p_l = b_{l+1}$ and not $p_l = b_l$. Thus, under the *LEF* criterion, each player chooses a position that maximises his payoff taking prices as given and the process reaches equilibrium prices that then clear the market. In order to implement a stable competitive equilibrium, bidders need to be *locally indifferent.* No player has an incentive to deviate and to bid for the position just above him. The player assigned to position k has to be *locally indifferent* between winning position $k-1$, paying his own bid and position k paying the next-highest bid. This implies that $\alpha_k x_k - \alpha_k p_k = \alpha_{k-1} x_k - \alpha_{k-1} p_{k-1}$, which gives the following set of bids $\boldsymbol{b} = (b_1, b_2, b_3)$:

$$\begin{aligned} & b_1 > b_2 \\ & b_2 = x_2 - \frac{\alpha_2}{\alpha_1} (x_2 - b_3^v) \\ & b_3 = x_3 \end{aligned} \tag{8}$$

The bid profile $\boldsymbol{b}$ corresponds to the lowest point in the set of competitive prices among all the *locally envy-free* equilibria. Even if truth-telling is not an equilibrium strategy of the GSP auction, the outcome is equivalent to the VCG solution.

Notations: Two equilibria, which we refer to as *Lower-Nash Equilibrium* (LE) and *Upper-Nash Equilibrium* (UE), reside at the boundaries of the *Nash* set. Each profile is denoted respectively by $\boldsymbol{b}^l = \left(b_i^l\right)_{i=1,2,3}$ and $\boldsymbol{b}^u = (b_i^u)_{i=1,2,3}$. We call the equilibrium achieved when restricting b_3 as being equal to x_3 the *Dom-Nash* equilibrium denoted by $\boldsymbol{b}^d = \left(b_i^d\right)_{i=1,2,3}$ and denote the VCG outcome by $\boldsymbol{b}^v = (b_i^v)_{i=1,2,3}$. Then, from a revenue perspective, any *symmetric* equilibrium (or *locally envy-free* equilibrium) induces a revenue at least equal to the *VCG* revenue (e.g., Feldman et al. (2011), Lucier et al. (2012)). As a result, if we denote by R^l and R^u the auctioneer's revenues achieved by $\boldsymbol{b}^l$ and $\boldsymbol{b}^u$, respectively we have that $R^l \leq R^v \leq R^u$.

3 Equilibrium and revenue

3.1 Collusive equilibria

Consider that the *bidding agency* acts as a credible banker to which bidders entrust their expected cash surplus from colluding *efficiently.* The *agency* computes individual contributions conditional on the efficient *Nash* allocation deduced from the set (7). We assume that contributions are *allocation-independent,* computed *ex ante* and, thus, that they are not altered by potential deviations.

Let $\theta = \frac{\alpha_2}{\alpha_1} \in [0,1]$ denotes the ratio of clicks between positions 2 and position 1 and the *non-cooperative* bid profiles to be denoted by $\boldsymbol{b}^g = (b_i^g)_{i=1,2,3}$ with $g = l, v, u$.

Fact 1. *The more positions are substitutes the less robust collusive agreements are.*

In order to remain optimal, non-cooperative bids decrease in θ. If the substitutability between both positions becomes perfect, bidders should bid less as the need to outbid competitors to win the highest position decreases. However, when coordination is active, if the substitutability between positions increases, incentive compatibility constraints become more stringent. Profits from cheating and aiming at second position of the player with the lowest valuation increases in θ. To see this, recall that from relations (4) and (5), incentive compatibility constraints for this member are given by $IC_3 : \tau \geq \alpha_1 \left(x_3 - b_1^{\mathcal{N}}\right) \equiv \tilde{\pi}_3(.)$ and $IC_3' : \tau \geq \alpha_2 \left(x_3 - b_2^{\mathcal{N}}\right) \equiv \tilde{\pi}_3(.)$. The player's payoff from deviating to position one is independent of θ. However, his payoff is strictly increasing in θ in the case of a deviation to the second position. The *bidding agency* is under a constraint to release the second-highest valuing member's bid in order to maintain the ranking, which in turn makes the internal competition difficult to contain.

Consider contract $\gamma_i = \left(0, \left\{b_i^{\mathcal{N}}, p_i^{\mathcal{N}}\right\}\right)$, where the *agency* is benevolent. It maximises advertisers' surplus and acts as a social planner by not extracting benefits from the collusive surplus. We can begin by stating that in the absence of commission fees, the *agency* implements the most subversive outcome in relation to the seller's revenue. It coordinates advertisers over the smallest price equilibrium, at which they capture the entire market rent.

Proposition 1. *Suppose there is no commission fee. Then, for $\theta \geq \frac{b_2^g}{3x_3 - b_3^g}$ contract $\gamma_i = \left(0, \left\{b_i^{\mathcal{N}}, p_i^{\mathcal{N}}\right\}\right)$ maximises individual profits and implements an efficient outcome, whereby all the rent is captured by advertisers.*

Proof. The proof follows from that of proposition 2 setting $\varepsilon = 0$. □

As in the benchmark, *cartel* members can envision multiple best-replies to their respective equilibrium collusive strategies, and the underlying incentive compatibility constraints allow for symmetric/asymmetric equilibria. This proposition states that if the degree of substitutability between positions is sufficiently high, the *agency* coordinates advertisers in a first-best outcome, which rules out the multiplicity of collusive equilibria. In equilibrium, individual payments are minimised and competition from the middle-valuing advertiser is just high enough to secure the second slot and to deter defections from the lowest-valuing advertiser. The rivalry between bidders is suppressed, thus blocking rent appropriation from the seller, which is fully captured by the advertisers.

Let us now assume that the *agency* seeks to extract surplus by charging advertisers a uniform commission fee, i.e., $\varepsilon > 0$. The set of compatible bids is relegated to appendix A and is given by relation (14). The following lemma provides conditions on the ratio of clicks θ for the competition between members of the coalition to be muted and will helps to discriminate among the set of compatible collusive bids.

Lemma 1. *A sufficient condition for $b_2^{\mathcal{N}} > 0$ is $\theta \geq \eta \equiv \frac{(1-\varepsilon)b_2^g}{3x_3 - (1-\varepsilon)b_3^g}$, and for $b_3^{\mathcal{N}} = 0$ is $\theta \leq \rho \equiv \frac{3x_1 + (1-\varepsilon)b_2^g}{3x_3 - (1-\varepsilon)b_3^g}$.*

Consider the optimal contract $\gamma_i^* = \left(\varepsilon, \left\{b_i^{\mathcal{N}}, p_i^{\mathcal{N}}\right\}\right)$ such that the corresponding profile $\boldsymbol{\mu}_{\mathcal{N}}^*$ defines the smallest efficient collusive equilibrium. That is, bidders are allocated in decreasing order of their valuations with τ maximised given both the bid profile $\boldsymbol{b}^g = (b_i^g)_{i=1,2,3}$ and the fee ε. The following proposition summarises the profit-maximising equilibrium bids implemented during the auction on behalf of each advertiser.

Proposition 2. *Suppose the* agency *charges a positive fee. Then,* $\gamma_i^* = \left(\varepsilon, \left\{b_i^{\mathcal{N}}, p_i^{\mathcal{N}}\right\}\right)$ *is an equilibrium contract in which the corresponding bid profile* $\boldsymbol{\mu}_{\mathcal{N}}^* = \left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$ *is monotonically non-decreasing in* ε *and decreasing in* $\boldsymbol{b}^g = (b_i^g)_{i=1,2,3}$ *with* $g = l, v, u$.

(i) When $\theta < \eta$ *and* $\varepsilon \leq \delta$*, the optimal allocation is inefficient with* $\boldsymbol{\mu}_{\mathcal{N}}^* = \left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$,

$$\begin{aligned} b_1^{\mathcal{N}} &> b_2^{\mathcal{N}} \\ b_2^{\mathcal{N}} &= b_3^{\mathcal{N}} = 0 \end{aligned}$$

(ii) When $\theta \in [\eta, \rho]$ *and* $\varepsilon \leq \delta$*, a constrained first-best is achieved with* $\boldsymbol{\mu}_{\mathcal{N}}^* = \left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$,

$$\begin{aligned} b_1^{\mathcal{N}} &= x_3 - \frac{(1-\varepsilon)}{3\alpha_1} \sum_{i=1}^{3} \alpha_i \left(b_{i+1}^g - b_{i+1}^{\mathcal{N}}\right) \\ b_2^{\mathcal{N}} &= \left(1 - (1-\varepsilon)\frac{\alpha_1}{\alpha_2}\right)\left(x_3 - \frac{(1-\varepsilon)}{3\alpha_2} \sum_{i=1}^{3} \alpha_i b_{i+1}^g\right) \\ b_3^{\mathcal{N}} &= 0 \end{aligned} \tag{9}$$

(iii) When $\varepsilon > \delta$*, internal bidder's rivalry cannot be constricted with* $\boldsymbol{\mu}_{\mathcal{N}}^* = \left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$,

$$\begin{aligned} b_1^{\mathcal{N}} &> b_2^{\mathcal{N}} \\ b_2^{\mathcal{N}} &= \frac{(2+\varepsilon)\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2} \alpha_i b_{i+1}^g - (1-\varepsilon)(\alpha_1 - \alpha_2) x_1}{(2+\varepsilon)\alpha_2 - (1-\varepsilon) 2\alpha_1} \\ b_3^{\mathcal{N}} &= \frac{(4-\varepsilon)\alpha_1\alpha_2 x_3 - (1-\varepsilon)(\alpha_1+\alpha_2)\sum_{i=1}^{2} \alpha_i b_{i+1}^g - x_1(\alpha_1 - \alpha_2)(3\alpha_2 - (1-\varepsilon)\alpha_1)}{\alpha_2\left((2+\varepsilon)\alpha_2 - (1-\varepsilon)2\alpha_1\right)} \end{aligned} \tag{10}$$

with

$$\delta = \frac{(1+\theta)\sum_{i=1}^{2} \alpha_i b_{i+1}^g - (1-\theta)(\alpha_1 - 3\alpha_2) x_1 - 4\alpha_2 x_3}{(1+\theta)\sum_{i=1}^{2} \alpha_i b_{i+1}^g - (1-\theta)\alpha_1 x_1 - \alpha_2 x_3} \tag{11}$$

Part (i) states that when positions differ too much, then the *bidding agency* implements the worst senario for the seller in terms of revenues but the final allocation is inefficient in the sense that bidder 2 and 3 are randomly assigned to positions.

Part (ii) shows that if the degree of substitutability between both positions is sufficiently high and the *bidding agency* is not too greedy, the game results in a constrained *low-price* outcome. In equilibrium, assigned members' bids are set equal to a quantity that is strictly below the valuation of the non-assigned member, who refrains from bidding. Then, bid delegation takes the form of an efficient coordination according to definition 2 which stabilizes the auction to a unique efficient allocation constrained by the fee level.

Finally, part (iii) shows that when $\varepsilon > \delta$, the *bidding agency* needs to break the non-assigned member's

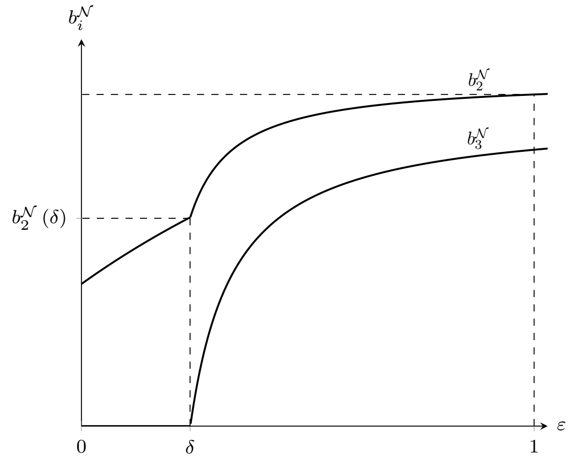


Figure 2: Collusive equilibrium bids as a function of ε.

incentives to defect if it were to expropriate a higher share of the spoils, but this is done at the expense of internal rivalry (see Figure 2, in which we have set $b_1^{\mathcal{N}} = b_2^{\mathcal{N}} + \epsilon$ as it affects neither payments nor revenues). This observation results from the following fact,

Fact 2. *The marginal loss in advertisers' profits due to an increase in the fee is compensated by a marginal increase in the coordinated bids.*

To see why, recall that individual payoff function is equal to:

$$\pi_i^{\mathcal{N}}\left(x_i; b_i^{\mathcal{N}}\right) = \alpha_k\left(x_i - b_{k+1}^{\mathcal{N}}\right) - \left(\alpha_k b_{k+1}^{g} - \alpha_k b_{k+1}^{\mathcal{N}}\right) + \frac{1-\varepsilon}{3}\sum_{j=1}^{2}\alpha_j b_{j+1}^{g} - \frac{1-\varepsilon}{3}\sum_{j=1}^{2}\alpha_j b_{j+1}^{\mathcal{N}}$$

it is thus obvious to see that $\frac{\partial^2 \pi_i}{\partial\varepsilon\partial b_i^{\mathcal{N}}} > 0$. Hence, greed on the part of the *bidding agency* generates more aggressive behaviour from members in order to maintain incentive aligned and to compensate the downward shifting of the collusive gain in terms of individual contributions.

Finally, from the benchmark we know that the *GSP* auction has a multiplicity of *Nash* equilibria. Without coordination, nothing guarantees a convergence to a *non-cooperative* solution which Pareto dominates any other solutions from advertisers' viewpoint. Proposition 2 also relates to the sustainability of such a solution insofar as it represents a limit case for any collusive mechanism. As highlighted by part (iii), the *bidding agency* is able to push the equilibrium collusive prices upward so that bidders reverse to *non-cooperative* behaviour.

Corollary 1. *The* non-cooperative *equilibrium is a sustainable collusive outcome with a uniform-ε redistribution rule. As $\varepsilon \mapsto 1$ the* bidding agency *coordinates bidders over the lowest* Nash *equilibrium bid profile $\boldsymbol{b}^l = \left(b_i^l\right)_{i=1,2,3}$.*

Proof. The result is immediate if we set $\varepsilon = 1$ into the equilibrium collusive bidding functions defined in relations (11). We obtain $b_2^{\mathcal{N}} = x_3 = b_2^l$ and $b_3^{\mathcal{N}} = x_1 - \frac{\alpha_1}{\alpha_2}(x_1 - x_3) = b_3^l$. The objective of the *agency* is thus now to maximise the function $BS = \sum_{i=1}^{3}\left(\alpha_i\left(x_i - b_{i+1}^{g}\right)\right)$ under the same *IC* constraints of the *non-cooperative* benchmark. This entails $\boldsymbol{b}^l$ as a natural equilibrium outcome. $\square$

Obviously, collusive payoffs are no lower than in their corresponding *non-cooperative* counterpart. However, by incrementally increasing its expropriation abilities, the *agency* is able to evict all other equilibria, so that the simplest incentive compatible collusive mechanism that is always feasible turns out to be the smallest *non-cooperative* outcome in which players set their optimal bids consistently with $\boldsymbol{b}^l = \left(b_i^l\right)_{i=1,2,3}$. If the *bidding agency* acts according to proposition 2 no bidder will find it profitable to leave the cartel. Then, if $\varepsilon = 1$, it can bid in two different ways. It can pick any bid vector compatible with the *Nash* equilibrium criterion so that it implements any *Nash* outcome without being interested in the bidders' individual welfare. Otherwise, it can decide to maximise the latter. By doing so, it picks the price vector that maximises each individual's profits, which is $\boldsymbol{b}^l = \left(b_i^l\right)_{i=1,2,3}$. This implies that in the limit case, individual payoffs from coordination are equal to the corresponding equilibrium payoffs with $\boldsymbol{b}^l$.

To summarise, our results show that the *bidding agency* can provide an efficient collusive scheme to which advertisers voluntarily comply and under which it can expropriate a positive share of the spoils without breaking incentives to collude. However, this cannot result in a full rent expropriation as shown by the last corollary. Within competition strictly increases in the management fee which implies a trade-off faced by the *bidding agency* when setting the management fee between surplus expropriation, advertisers' rivalry and low-price equilibrium: a higher fee level increases the *agency*'s revenue but drives up equilibrium prices, i.e., advertisers rivalry, and lowers joint profits.

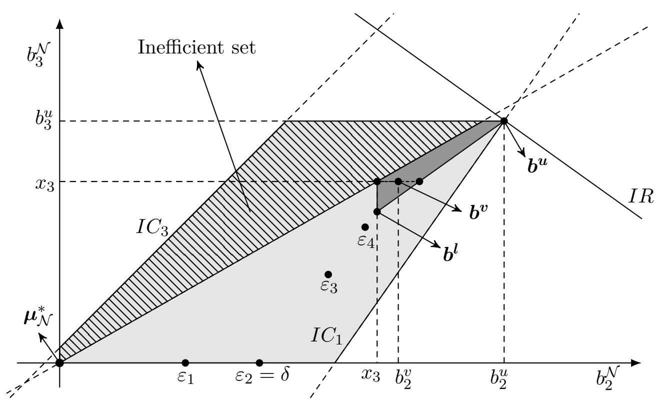


Figure 3: Set of collusive bids in light grey with $\varepsilon_0 = 0, \varepsilon_1 = 0.3, \varepsilon_2 = \delta, \varepsilon_3 = 0.6$ and $\varepsilon_4 = 0.9$.

Example 1. *Let the set of valuations be $\boldsymbol{x} = (5, 4, 3)$ and the number of clicks received by each position to be $\boldsymbol{\alpha} = (10, 8)$. Suppose that as an outside option, the* bidding agency *conjectures that bidders will play according to the bid profile $\boldsymbol{b}^u = (b_1^u, b_2^u, b_3^u)$, defined as the upper bound of each interval of equilibrium bids in (7). This gives $b_3^u = 4, b_2^u = 4.2$ and $b_1^u = 5$ and the corresponding collusive set is pictured in Figure 3. In Figure 3, we represent equilibrium outcomes for different fee level, ranging from a pure benevolent* agency *at $\varepsilon = 0$ to an* agency *that is an "absolute rent seeker" at $\varepsilon = 1$. In this environment $\delta \simeq 0.51$ and $\eta = 0.84$ so $\theta < \eta$. When the* agency *does not seek to extract rent, the equilibrium proposal $\boldsymbol{\mu}^*_{\mathcal{N}} = \left(b_1^{\mathcal{N}}, b_2^{\mathcal{N}}, b_3^{\mathcal{N}}\right)$ for $\varepsilon = 0$ is given by $b_3^{\mathcal{N}} = 0, b_2^{\mathcal{N}} = 0$ and $b_1^{\mathcal{N}} > 0$. Both players 2 and 3 refrain from bidding, which results in a suboptimal allocation (with both players being randomly assigned position two). This constitutes the worst scenario for the seller as the* non-cooperative *game results in a revenue of $R^u = \alpha_1 b_2^u + \alpha_2 b_3^u = 74$, whereas under bid coordination, revenues are $R_{\boldsymbol{\mu}_{\mathcal{N}}} = 0$.*

This corresponds to the standard low-revenue property *encountered in multi-object auctions with active collusion. As shown by the corollary, the collusive outcome converges to* $\boldsymbol{b}^l$ *as* $\varepsilon \mapsto 1$*. At this point,* $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_1^{\mathcal{N}}, b_2^{\mathcal{N}}, b_3^{\mathcal{N}}\right) = \left(b_2^{\mathcal{N}} + \varepsilon, 3, 2.5\right)$*, generating a revenue* $R_{\boldsymbol{\mu}_{\mathcal{N}}} = 50$*. Consider that the outside bid profile is set to* $\boldsymbol{b}^l = \left(b_1^l > b_2^l, x_3, x_1 - \frac{\alpha_1}{\alpha_2}(x_1 - x_3)\right)$*, then* $R^l = \alpha_1 b_2 + \alpha_2 b_3 = 50$ *and* $R_{\boldsymbol{\mu}_{\mathcal{N}}} = 50$ *if* $\varepsilon = 1$*.*

3.2 The auctioneer and the *agency* revenue

In this subsection we first characterise the optimal level of management fee that solves the tension between surplus expropriation and internal level of competition, and then examine its implications on the search engine's revenue.

Lemma 2. *The fee threshold* δ *is the point at which the lowest member is indifferent to being active or staying inactive. It is a non-decreasing function of the* Nash *bid profile* $\boldsymbol{b}^g = (b_i^g)_{i=1,2,3}$ *with* $g = l, v, u$*.*

The lemma means that the threshold δ defined in (11) is such that when $\varepsilon > \delta$ the *agency* is constrained to set the lowest-valuing member's bid to $b_3^{\mathcal{N}} > 0$, which increases within-competition, if it were to sustain the efficient ranking and the coordination. Thus, when $\varepsilon > \delta$ the *first-best outcome* property of definition 2 cannot be maintained.

Then, $\forall \varepsilon \in [0, \delta]$ the equilibrium strategy profile is characterised by the set (9) and the *bidding agency*'s profit is equal to the following quantity:

$$\Gamma_{\mathcal{N}}^{'} = \varepsilon\left(\alpha_1\left(b_2^g - b_2^{\mathcal{N}}\right) + \alpha_2\left(b_3^g - b_3^{\mathcal{N}}\right)\right) = \frac{3\alpha_2\left(R^g - \alpha_1 x_3\right)}{3\alpha_2 - (1-\varepsilon)\alpha_1}\varepsilon$$

and $\forall \varepsilon \in (\delta, 1)$, by the equilibrium bids (11), it takes the following quantity:

$$\Gamma_{\mathcal{N}}^{''} = \frac{3\alpha_2\left(R^g + (\alpha_1 - \alpha_2)x_1 - 2\alpha_1 x_3\right)}{(2+\varepsilon)\alpha_2 - (1-\varepsilon)2\alpha_1}\varepsilon$$

with $R^g := (\alpha_1 b_2^g + \alpha_2 b_3^g)$. The *bidding agency*'s profit function is thus a piecewise function with unique intercept at $\varepsilon = \delta$, and the following proposition identifies its optimal pricing strategy.

Proposition 3. *The* bidding agency*'s revenue is non-monotonic in* ε*: it is concave* $\forall \varepsilon \leq \delta$ *and convex* $\forall \varepsilon > \delta$*. An optimal incentive-compatible fee at which an* agency *maximises its revenue is set equal to the point* $\varepsilon^* = \delta$*.*

The optimal policy for the *agency* is then to set a fee that makes the lowest-valuing member of the coalition indifferent between staying out the auction or breaking the collusion. For sufficiently low level of fee, biasing the redistribution of the rent is possible as it can be achieved without deterring the collusive incentives. Even if bids are monotonically increasing in ε, it is able to increase its profits as the increase in fees sufficiently compensates for the increase in collusive prices. However, the threshold δ defines the critical point at which a greedy attitude becomes pervasive. The loss in the bidders' surplus needs to be compensated for by an increase in the respective collusive bids, which implies a shift in the *bidding agency*'s profits along with a higher level of internal competition (see Figure 4). The level of expropriation from the *bidding agency* makes the incentive conditions more stringent for the bidders, preventing them from showing any self-interest in the cartel formation.

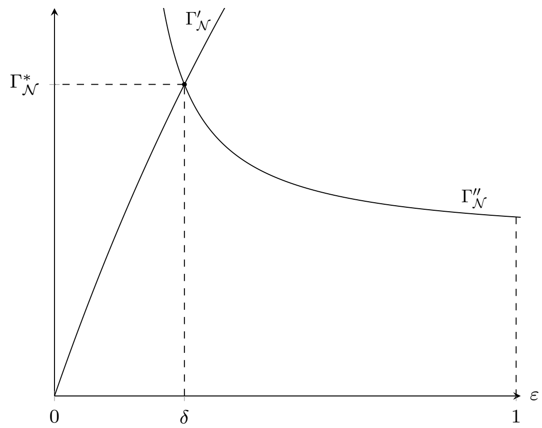


Figure 4: The *bidding agency*'s profits as a function of ε

Suppose that the search engine is able to affect the click-through rate couple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and was to set them arbitrarily. From proposition 3, we know that the *bidding agency* has no incentives to set $\varepsilon > \delta$; thus, it is sufficient to focus on the bid functions of relation (9). Taking the corresponding equilibrium bid profile $\boldsymbol{\mu}_{\mathcal{N}}$ and rearranging the expressions so that it becomes a function of θ. We obtain,

$$b_2^{\mathcal{N}}(\theta) = \frac{3\theta x_3 - (1-\varepsilon)(b_2^g + \theta b_3^g)}{3\theta - (1-\varepsilon)}$$

If we suppose that $\alpha_1 \mapsto \infty$ so that $\theta = 0$, then from lemma 1 we have that $b_2^{\mathcal{N}} = b_3^{\mathcal{N}} = 0$. Hence, the auctioneer's revenues would fall to zero. If now the auctioneer sets $\alpha_1 = \alpha_2$ so that $\theta = 1$. Then, from proposition 3 it is optimal for the *bidding agency* to set $\varepsilon^* = \delta$ and we obtain,

$$\begin{aligned} \varepsilon^* &= \frac{2(b_2^g + b_3^g - 2x_3)}{2(b_2^g + b_3^g) - x_3} \\ b_2^{\mathcal{N}} &= \frac{1}{2} x_3 \end{aligned}$$

As a result, the auctioneer's revenue cannot be higher than in the corresponding equilibrium profile $\boldsymbol{b}^l$ and we can therefore make the following claim,

Claim 1. *An auctioneer cannot use the ratio θ in order to induce levels of revenue higher than those induce by the lowest* Nash *equilibrium bid profile $\boldsymbol{b}^l$.*

The next proposition summarises the revenues implications of an efficient bid delegation.

Proposition 4. *The auctioneer's revenue is monotonically increasing in ε and the maximum quantity of surplus it can extract is no higher than under the corresponding non-cooperative equilibrium bid profile $\boldsymbol{b}^l$.*

Proposition 4 states that greed on the part of *bidding agency* benefits to the search engine and revenues from the auction are upper bounded by the one achieved in the smallest *non-cooperative* equilibrium in price. As highlighted by Fact 2, $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$ strictly increases in ε. The *bidding agency* needs to compensate for the loss by pushing collusive prices forward in order to maintain incentives aligned, which mechanically benefits to the search engine (see Figure 5). If the *bidding agency* sets the maximum fee, i.e., $\varepsilon = 1$, $\boldsymbol{\mu}_{\mathcal{N}}$ converges to $\boldsymbol{b}^l$, and so does the auctioneer's revenue.

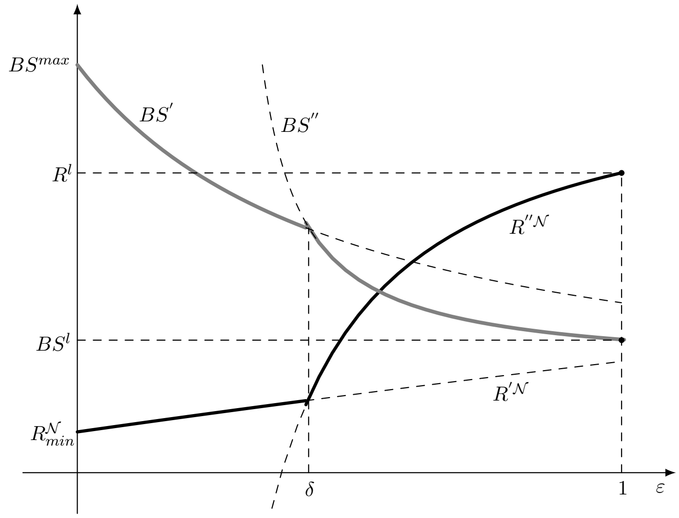


Figure 5: Search engine's revenue $R^{\mathcal{N}}$, bidders' surplus BS and bidders' surplus under $\boldsymbol{b}^l$ as a function of ε

More generally, propositions 3 and 4 offer insights about the incentives of the search engine to not acting against bid delegation services. Its revenues are based on maintaining a long run strong market position, which is derived from advertisers' participation incentives. While deterring rent extraction to the advertisers' benefit, the optimal policy of the *bidding agency* secures a lower bound in the search engine's surplus and ensures allocative efficiency, which should ultimately generates near-optimal revenues on the long-run (see Bulow and Klemperer (1996) and Roughgarden and Sundararajan (2007)).

3.3 Substitutability between positions and competition

In this section, we compare the link between the click ratio θ as a measure of the degree of substitutability of positions and the *demand-reduction* phenomenon observed in the *GSP* auction and, more generally, in multi-object auctions. Bid shading occurs in situations in which players' bids affect the price they have to pay at the end of the auction which relies on the assumption that bidders have multi-unit demands. As a result, this should disappear whenever the same player asks for one object (one position) at the most.

In the *GSP* auction, advertisers can ask for only one position. Notice the special feature in the competition nature and equilibrium predictions suggested by the bid profile $\boldsymbol{b}^v$. In equilibrium, it is optimal for each player to bid strictly below their own valuation (except for the non-assigned player). In fact, this phenomenon can be extended or broken down with the differentiation in clicks between positions. This points to the existence of a close link between *ctr* and the nature of the competition in the *GSP* auction.

Remark 2. *The relation between the ratio $\theta = \frac{\alpha_2}{\alpha_1}$, the equilibrium non-cooperative bids and payments is summarized in the following observations:*

- (i) *As $\theta \mapsto 0$, bid shading disappears. The game converges essentially to a standard second-price auction.*
- (ii) *As $\theta \mapsto 1$, bid shading increases. The game essentially converges to a* Bertrand *competition.*

Take the set of *Nash* equilibrium bids in (7) and observe that the bids are convex combinations weighted by the ratio θ. Without loss of generality, let us consider the refinement whereby in the *non-cooperative* game the non-assigned bidder (bidder 3) uses a dominant strategy, i.e., $b_3 = x_3$. We obtain $b_1 \in \left[x_2 - \frac{\alpha_2}{\alpha_1}(x_2 - x_3)\,;\,\bar{x}\right]$, $b_2 \in \left[x_3\,;\,x_1 - \frac{\alpha_2}{\alpha_1}(x_1 - x_3)\right]$ and $b_3 = x_3$. It is now obvious to see that when $\theta = 0$, i.e., the first position is the only object worth winning (the most attractive one), bids are bounded by the next-to-top and the next-to-bottom valuations. That is we obtain $b_1 \in [x_2\,;\,\bar{x}]$, $b_2 \in [x_3\,;\,x_1]$ and $b_3 = x_3$. The link is even closer at the *symmetric* equilibrium, where indeed, bidding one's valuation is the only possible bid, as it is in standard second-price auctions.

Now, for (*ii*), note that, as $\theta = 1$, both positions are substitutes and the middle-valuing player's equilibrium bid is equal to $\max\{b_3, x_3\}$. There is no opportunity for the highest-valuing player to undercut him and the lowest-valuing advertiser sets the market-clearing price to his private valuation. We have that $b_1 \in [x_3\,;\,\bar{x}]$, $b_2 = x_3$ and $b_3 = x_3$. This situation highlights the analogy with a Bertrand competition and the well-known *demand-reduction* phenomenon in standard multi-object auctions[18]. The lowest-valuing bidder's valuation determines the shape of the market-clearing price. Nevertheless, there is still an opportunity for the auctioneer to make the top position the only one worth having by destroying incentives to shade in the *non-cooperative* play.

Coordinated bids: However, there are no clear tendencies for the link between θ and the nature of the competition with an active cartel.

Remark 3. *The relation between the ratio $\theta = \frac{\alpha_2}{\alpha_1}$, the equilibrium coordinated bids and payments is summarized in the following observations:*

(*i*) *As $\theta \mapsto 0$, bid shading increases. The game essentially converges to a collusive implementation in a single unit second-price auction.*

(*ii*) *As $\theta \mapsto 1$, bid shading persists but there is no clear tendency in its magnitude. It depends on the* agency*'s objective and on the equilibrium bid profile* $\boldsymbol{b}$ *fixed as outside option.*

If the substitutability between positions was to decrease, that is $(\alpha_1 - \alpha_2)$ increases, then if the *bidding agency* behaves as in proposition 2 and set $\varepsilon = \delta$, the coordinate bids of player 2 would equal zero and the link would be reversed. As a result, we would nest the standard result of collusion in second-price auctions with multi-unit objects and bid shading is exacerbated. However, if $\theta = 1$, which means that $(\alpha_1 - \alpha_2)$ increases, we obtain $b_2^{\mathcal{N}} = \frac{3x_3 - (1-\varepsilon)(b_2 + b_3)}{2+\varepsilon}$ and $b_3^{\mathcal{N}} = 0$, which depends on the bid profile $\boldsymbol{b}$. If we assume that the *bidding agency* seeks to maximise the bidders surplus, then it coordinates the outside option towards $\boldsymbol{b}^l = \left(b_2^l = x_3, b_3^l = x_1 - \frac{\alpha_1}{\alpha_2}(x_1 - x_3)\right)$ and when $\varepsilon = \delta$ we get $b_2^{\mathcal{N}} = \frac{1}{2}x_3$ and $b_3^{\mathcal{N}} = 0$.

For $\varepsilon > \delta$, then $b_2^{\mathcal{N}} = x_3 = b_3^{\mathcal{N}}$. Thus, if $(\alpha_1 - \alpha_2)$ increases, then for $\varepsilon = \delta$ the bid placed by player 2 equals half the valuation of the lowest-valuing bidder, whereas $\forall \varepsilon > \delta$ $b_2^{\mathcal{N}}$ increases to x_3. As a result, coordinated bids increase in θ and in the degree of substitutability between positions as highlighted by fact 1.

[18] See for instance Engelbrecht-Wiggans and Kahn (1998), Ausubel et al. (2014).

3.4 Link with the LEF criterion and VCG outcome

In this section we discuss whether the *Vickrey-Clarke-Groves* (*VCG*) solution can be compatible and implemented by a rent-seeking *bidding agency*. It is shown that, if we scale the analysis over the spirit of standard competitive equilibrium analysis by using the LEF criterion in the present framework, then a profitable way to collude does not exist except by achieving the *VCG* outcome. In other words, if deviations are managed by members according to the argument of *local indifference* then the *GSP* auction becomes collusion-proof and the equivalence with the *Vickrey-Clarke-Groves* solution is restored and unique.

What would the effect be if we were to apply the LEF *stability condition* to the collusive game, selecting from the collusive equilibria the one that respects this criterion? In other words, let us consider the LEF criterion as a *stability condition* instead of the expression of a *Walrasian tatônnement* process (as in Börgers et al. (2013)), the latter being necessarily related to a *non-cooperative* analysis. Assume still that, in the case of deviation from the collusive agreement, the defector pays his individual contribution computed *before the targeting auction starts*, where individual contributions are computed unconditionally on deviations and are based on an assortative assignment. In addition, assume still that the *bidding agency* makes its transfer payment after the main auction. The *incentive compatibility* conditions are now given by the following relations:

$$\pi_k^{\mathcal{N}}\left(b_k^{\mathcal{N}},\boldsymbol{b}_{-i}^{\mathcal{N}},x_k\right)-\tilde{\pi}_k\left(\tilde{b}_k,\boldsymbol{b}_{-i}^{\mathcal{N}},x_k\right) \quad \geq \quad 0 \tag{12}$$

where $\forall k \in \mathcal{K}$:

$$\tilde{\pi}_k\left(\tilde{b}_k,.\right) \quad = \quad \alpha_{k-1}\left(x_k-b_k^{\mathcal{N}}\right)-\left(\alpha_k b_{k+1}-\alpha_k b_{k+1}^{\mathcal{N}}\right) \tag{13}$$

This condition says that when the player assigned to position k intends to defect from the collusive agreement and contemplates position $k-1$, then he expects to pay the same price as the player assigned to this position. Hence, we give the mechanism a semblance of symmetry in the collusive equilibrium conditions and this results in each member bidding the *non-cooperative symmetric* equilibrium prices.

Proposition 5. *Under the locally envy-free condition and a* uniform-ε *redistribution a cartel with three players can do no better than achieve the* VCG *outcome. That is:* $\boldsymbol{\mu}_{\mathcal{N}}^{*}=\left(b_1^{\mathcal{N}}>b_2^{\mathcal{N}}\,;\,\frac{P_1^v}{\alpha_1}\,;\,x_3\right)=\boldsymbol{b}^v$.

The underlying justification of the choice of the respective *non-cooperative* bids is as follows. In the *non-cooperative* game, if bidders pick, from the set of competitive prices, the one that maximises their individual profits, then the *LEF* criterion results in them implementing the *VCG* outcome. In our present context, the *bidding agency* assumes the role of a *mediator* that manages individual choices. Then, if it were to act as a benevolent *mediator*, under the scope of the competitive equilibrium it would choose the vector of *non-cooperative* prices that maximises the bidders' surplus. The purpose of proposition 5 is to show that the *bidding agency* cannot implement a compensation policy that is compatible with a *first-best* collusive outcome under this stability criterion. It is interesting to note that bidding functions are now completely independent of the *agency*'s compensation policy. Hence, the key idea of this proposition is that the *GSP* auction, together with an efficient collusion, implements, in equilibrium, the *VCG*-equivalent outcome as a result of explicit coordination. Sponsored search markets entail a tri-partite structure in which third parties bid on behalf of different advertisers. As a result, our model offers a consistent justification as to why bidders behave according to the *LEF* refinement when bid delegation is involved.

4 Concluding remarks

Bidding agencies have emerged to help advertisers internalize their costs of knowledge acquisition and to absorb the level of competition they themselves would create by acting non-cooperatively. To the best of our knowledge, this study is the first to offer a specific examination of an explicit bid coordination implemented by a *bidding agency* at a *GSP* auction.

Specifically, the paper proposes a characterisation of the bid profiles that can be implemented in a two-position game involving three players . Bidders can efficiently collude via the *bidding agency*, but this adversely affects *GSP* auction revenues. This study also proposes a closed-form expression for the optimum fee level that can be implemented and highlights the non-monotonicity of the agency's revenues. It is argued that greedy behaviour on the part of the *bidding agency* is detrimental to the sustainability of an efficient collusion but that it manages to coordinate advertisers at the least efficient *Nash* equilibrium. Our results suggest that an *agency* can deliver a profitable bid delegation service, and that its presence in the sponsored search market increases the advertisers' surplus and contributes to market efficiency. We believe that our model offers a stylized justification of both the prevalence of the tri-partite market structure and of the compensation policies based on a fixed fee that prevail in this market.

The auctioneer is passive, which raises the question as to what it should do and if it it is in its interests to prevent collusion in the auction. We argue that it may not be in the auctioneer's interests to deter collusion if it takes a long-run perspective. A search engine's revenues are based on maintaining a strong market position, which is derived from advertisers' feelings that they are obtaining a high surplus. A one-shot auction might be greatly impaired by delegation but, in the long run, in line with the reasoning of Bulow and Klemperer (1996) and Roughgarden and Sundararajan (2007), the auction should generate near-optimal revenues. Thus, from our perspective it should be in the interests of the auctioneer to maintain this structure and even cooperate with *bidding agencies* as shown by corollary 1 and proposition 4. Finally, the non-cooperative *VCG* solution is uniquely determined through the process of explicit coordination, which offers an intuitive and consistent justification when implemented non-cooperatively. We conclude by commenting on the assumptions made in our analysis and look at possible extensions of our work.

Our analysis rests on four main assumptions. First, we do not consider the possibility of repeated interactions, as we deal solely with the one-shot *GSP* auction game. Second, the *agency* occupies a monopoly position and does not have to face an adverse selection issue when implementing the collusive device. Third, once a potential member rejects the proposed contract then the cartel is not formed. Finally, individual contributions and the lump-sum transfer are computed and redistributed uniformly among members and are independent of individual deviations. In practice, the *GSP* auction is automatically triggered each time a search query is entered by a consumer, and this may favour tacit collusion as bidders have opportunities to compete against the same set of competitors. Our intuition is that the outcome will remain the same as the *bidding agency* can implement the additional threat of triggering non-cooperative bidding once a defection occurs. An interesting consideration here is the possibility of implementing a bid-rotation mechanism as in McAfee and McMillan (1992).

Multiple bidding agencies and incentives for the seller. Even if multiple *agencies* bidding in the name of different advertisers during the same set of keyword auctions appears rarely, an extension would be to integrate a competitive stage that includes different third parties and assess the effect of competition for the auction performance. This would introduce additional strategic scope and might act in favour of an increase in the level of competition with respect to our results. For instance, what would the outcome be of introducing an oligopoly of *agencies* that compete *à la* Bertrand by offering the lowest compatible fee?

Compensation schedules. The uniform fee and uniform redistribution of spoils assumptions seem to be good approximations of *agency* practices. They levy a fee, set *ex-ante*, on the revenues generated by each click to each advertisers. We believe that our model, therefore, makes sense in relation to the equal redistribution assumption. However, the model involves taxation on the collusive spoils that does not take into account individual members' identity and importance. Further extensions mays consider the case of a redistribution schedule contingent to each firm's contribution to the value of the coalition.

A Appendix

Set of compatible collusive bids

We seek to construct a collusive equilibrium bid profile that is compatible with the incentive compatibility constraints defined in (4) and (5) and with the individual rationality constraint defined by the relation (6). We set $\theta = \frac{\alpha_2}{\alpha_1}$.

From player 2's incentive compatibility constraint, we know he will not deviate to position 1 if the following relation is satisfied,

$$\alpha_2 \left(x_2 - b_3^{\mathcal{N}}\right) - \omega_2(.) + \frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} \geq \alpha_1 \left(x_2 - b_1^{\mathcal{N}}\right) - \omega_2$$

This relation gives the following conditions:

$$\begin{aligned} b_3^{\mathcal{N}} &\leq \frac{1}{(4-\varepsilon)\,\alpha_2}\left((1-\varepsilon)\sum_{i=1}^{2} p_i - (1-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} + 3\alpha_1 b_1^{\mathcal{N}} - (\alpha_1 - \alpha_2)\,3x_1\right) \\ b_1^{\mathcal{N}} &\geq \frac{1}{3\alpha_1}\left((\alpha_1 - \alpha_2)\,3x_2 - (1-\varepsilon)\sum_{i=1}^{2} p_i - (1-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} + (4-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right) \end{aligned}$$

giving the first lower bound for player 1 and an upper bound for player 3. Now from player 1's incentive compatibility constraint, we know that if he does not want to swap his position for position 2, then it should be the case that,

$$\alpha_1 \left(x_1 - b_2^{\mathcal{N}}\right) - \omega_1(.) + \frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} \geq \alpha_2 \left(x_1 - b_3^{\mathcal{N}}\right) - \omega_1(.)$$

This gives the following relations:

$$\begin{aligned}
b_2^{\mathcal{N}} &\leq \frac{1}{(4-\varepsilon)\,\alpha_1}\left((\alpha_1-\alpha_2)\,3x_1+\sum_{i=1}^{2}p_i+(2+\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right)\\
b_3^{\mathcal{N}} &\geq \frac{1}{(2+\varepsilon)\,\alpha_2}\left((4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}}-\sum_{i=1}^{2}p_i-(\alpha_1-\alpha_2)\,3x_1\right)
\end{aligned}$$

giving the first upper bound for player 2 and a lower bound for player 3. Finally, if we examine player 3's incentive compatibility constraint, we obtain,

$$\begin{aligned}
\frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} &\geq \alpha_1\left(x_3-b_1^{\mathcal{N}}\right)\\
\frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} &\geq \alpha_2\left(x_3-b_2^{\mathcal{N}}\right)
\end{aligned}$$

giving respectively:

$$\begin{aligned}
b_1^{\mathcal{N}} &\geq x_3-\frac{(1-\varepsilon)}{3\alpha_1}\sum_{i=1}^{2}\omega_i(.)\\
b_2^{\mathcal{N}} &\geq \frac{1}{3\alpha_2-(1-\varepsilon)\,\alpha_1}\left(3\alpha_2 x_3-(1-\varepsilon)\sum_{i=1}^{2}p_i+(1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right)
\end{aligned}$$

the second lower bounds for player 1 and 2. The participation constraint equals $\sum_{i=1}^{2} p_i \geq \sum_{i=1}^{2}\alpha_i b_{i+1}^{\mathcal{N}}$, which implies $b_2^{\mathcal{N}} \leq b_2$ and $b_3^{\mathcal{N}} \leq b_3$. The above inequalities result in the following set of equilibrium bids profiles $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$:

$$\begin{aligned}
b_1^{\mathcal{N}} &\in [\mathcal{A};\,\bar{x}]\\
b_2^{\mathcal{N}} &\in [\mathcal{B};\,\mathcal{C}]\\
b_3^{\mathcal{N}} &\in \left[\max\left\{0\,;\,\frac{1}{(2+\varepsilon)\,\alpha_2}\left((4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}}-(1-\varepsilon)\sum_{i=1}^{2}p_i-(\alpha_1-\alpha_2)\,3x_1\right)\right\}\,;\,\mathcal{D}\right]
\end{aligned}\tag{14}$$

in which,

$$\begin{aligned}
\mathcal{A} &\equiv \max\left\{x_3-\frac{(1-\varepsilon)}{3\alpha_1}\sum_{i=1}^{2}\omega_i(.)\,;\,\frac{1}{3\alpha_1}\left((\alpha_1-\alpha_2)\,3x_2-(1-\varepsilon)\sum_{i=1}^{2}p_i-(1-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}}+(4-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right)\right\}\\
\mathcal{B} &\equiv \max\left\{0\,;\,\frac{1}{3\alpha_2-(1-\varepsilon)\,\alpha_1}\left(3\alpha_2 x_3-(1-\varepsilon)\sum_{i=1}^{2}p_i+(1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right)\right\}\\
\mathcal{C} &\equiv \frac{1}{(4-\varepsilon)\,\alpha_1}\left((\alpha_1-\alpha_2)\,3x_1+\sum_{i=1}^{2}p_i+(2+\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right)\\
\mathcal{D} &\equiv \frac{1}{(4-\varepsilon)\,\alpha_2}\left((1-\varepsilon)\sum_{i=1}^{2}p_i-(1-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}}+3\alpha_1 b_1^{\mathcal{N}}-(\alpha_1-\alpha_2)\,3x_1\right)
\end{aligned}$$

Proof of lemma 1

Set $b_1^{\mathcal{N}}$ so that neither player 2 nor player 3 has incentives to deviate to the top position. The incentive compatibility conditions are:

$$\begin{array}{rcll}
(1-\varepsilon)\left(\alpha_1 b_2^g+\alpha_2 b_3^g\right)-3\alpha_2 x_3-(1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}+(3\alpha_2-(1-\varepsilon)\,\alpha_1)\,b_2^{\mathcal{N}} & \geq & 0 & IC_3\\
3\alpha_2 x_2+(1-\varepsilon)\left(\alpha_1 b_2^g+\alpha_2 b_3^g\right)-(4-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}+(1-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} & \geq & 0 & IC_2\\
3\alpha_1 x_1+(1-\varepsilon)\left(\alpha_1 b_2^g+\alpha_2 b_3^g\right)-(1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}+(4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} & \geq & 0 & IC_1\\
(\alpha_1-\alpha_2)\,3x_1+(1-\varepsilon)\left(\alpha_1 b_2^g+\alpha_2 b_3^g\right)+(2+\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}-(4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} & \geq & 0 & IC_1^{'}
\end{array}$$

where IC_3 is player 3's incentives to deviate for position 2, IC_2 is the player 2's incentives to occupy no position, IC_1 is player 1's incentives to occupy no position and $IC_1^{'}$ is player 1's incentives to occupy position 2. It can be observed that IC_3 implies both IC_2 and IC_1. Now, assume that $b_2^{\mathcal{N}}=b_3^{\mathcal{N}}=0$, we have:

$$\begin{array}{rcll}
(1-\varepsilon)\left(\alpha_1 b_2^g+\alpha_2 b_3^g\right)-3\alpha_2 x_3 & \geq & 0 & IC_3\\
(\alpha_1-\alpha_2)\,3x_1+(1-\varepsilon)\left(\alpha_1 b_2^g+\alpha_2 b_3^g\right) & \geq & 0 & IC_1^{'}
\end{array}$$

Thus, these bids are compatible if the following is true: $(1-\varepsilon)\left(b_2^g+\theta b_3^g\right)-3\theta x_3\geq 0$ and $(1-\theta)\,3x_1+(1-\varepsilon)\left(b_2^g+\theta b_3^g\right)\geq 0$: that is, if $\theta\leq\eta\equiv\frac{(1-\varepsilon)b_2^g}{3x_3-(1-\varepsilon)b_3^g}$ or if $\theta\leq\rho\equiv\frac{3x_1+(1-\varepsilon)b_2^g}{3x_3-(1-\varepsilon)b_3^g}$, where the first inequality implies the second. Thus, $b_2^{\mathcal{N}}=b_3^{\mathcal{N}}=0$ are compatible bids if $\theta\leq\eta$.

Proof of proposition 2

Since, $b_1^{\mathcal{N}}$ deters any deviation for the top position, it suffices to consider the adjacent deviations. Then, according to lemma 1, to ensure an efficient assignment we impose $\theta\in[\eta,\rho]$ so that $b_2^{\mathcal{N}}>0$ and maintain the bid $b_3^{\mathcal{N}}=0$. Both constraints are,

$$\begin{array}{rcll}
(1-\varepsilon)\displaystyle\sum_{i=1}^{2}\alpha_i b_{i+1}^g-3\alpha_2 x_3+(3\alpha_2-(1-\varepsilon)\,\alpha_1)\,b_2^{\mathcal{N}} & \geq & 0 & IC_3\\
(\alpha_1-\alpha_2)\,3x_1+(1-\varepsilon)\displaystyle\sum_{i=1}^{2}\alpha_i b_{i+1}^g-(4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} & \geq & 0 & IC_1^{'}
\end{array}$$

since $x_3<x_1$ if player 3's incentive to occupy position 2 is binding, it should also be the case for player 1. Thus, we are left with the constraint IC_3 in equilibrium which implies that,

$$(1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g-3\alpha_2 x_3+(3\alpha_2-(1-\varepsilon)\,\alpha_1)\,b_2^{\mathcal{N}}=0$$

That is,

$$b_2^{\mathcal{N}}=\frac{1}{3\alpha_2-(1-\varepsilon)\,\alpha_1}\left(3\alpha_2 x_3-(1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g\right)$$

Notice that, $b_2^{\mathcal{N}} < x_3$ since we can re-write the last expression as,

$$\begin{aligned} b_2^{\mathcal{N}} &= \frac{1}{3\alpha_2 - (1-\varepsilon)\,\alpha_1}\left(3\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g\right) \\ &= x_3 - \frac{(1-\varepsilon)}{(1-\varepsilon)\,\alpha_1 - 3\alpha_2}\left(\alpha_1\left(x_3 - b_2^g\right) - \alpha_2 b_3^g\right) \\ &= x_3 - \lambda\left(\alpha_2 x_3 - \alpha_2 b_3^g - \alpha_2 x_3 + \alpha_1 x_3 - \alpha_1 b_2^g\right) \\ &= x_3 - \lambda\left(\pi_3 + (\alpha_1 - \alpha_2)\, x_3 - \alpha_1 b_2^g\right) \end{aligned}$$

and that $IC_1^{'}$ is satisfied with such bid,

$$\begin{aligned} & (\alpha_1 - \alpha_2)\, 3x_1 + (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g - (4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} \\ \geq\ & (\alpha_1 - \alpha_2)\, 3x_1 + (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g - (4-\varepsilon)\,\alpha_1 x_3 \\ \geq\ & (\alpha_1 - \alpha_2)\, 3x_3 + (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g - (4-\varepsilon)\,\alpha_1 x_3 \\ \geq\ & (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g - (3\alpha_2 - (1-\varepsilon)\,\alpha_1)\, x_3 \geq 0 \end{aligned}$$

Now, consider the case in which $b_2^{\mathcal{N}} > 0$ and $b_3^{\mathcal{N}} > 0$. This is only possible towards ε and assume that there exists a threshold $\delta \in [0, 1]$ so that if $\varepsilon > \delta$ then $b_2^{\mathcal{N}} > 0$ and $b_3^{\mathcal{N}} > 0$. Both constraints equal,

$$\begin{aligned} (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g - 3\alpha_2 x_3 - (1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}} + (3\alpha_2 - (1-\varepsilon)\,\alpha_1)\, b_2^{\mathcal{N}} &\geq 0 \quad IC_3 \\ (\alpha_1 - \alpha_2)\, 3x_1 + (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g + (2+\varepsilon)\,\alpha_2 b_3^{\mathcal{N}} - (4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} &\geq 0 \quad IC_1^{'} \end{aligned}$$

Both constraints are binding at the optimum, otherwise the *bidding agency* can decrease $b_2^{\mathcal{N}}$ and $b_3^{\mathcal{N}}$ accordingly. We obtain,

$$\begin{aligned} b_2^{\mathcal{N}} &= \frac{1}{3\alpha_2 - (1-\varepsilon)\,\alpha_1}\left(3\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g + (1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right) \\ b_3^{\mathcal{N}} &= \frac{1}{(2+\varepsilon)\,\alpha_2}\left((4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^g - (\alpha_1 - \alpha_2)\, 3x_1\right) \end{aligned}$$

Plugging the expression of $b_2^{\mathcal{N}}$ in that of $b_3^{\mathcal{N}}$ results in the following relation,

$$b_3^{\mathcal{N}} = \frac{x_1\,(\alpha_1 - \alpha_2)\left((1-\varepsilon)\,\alpha_1 - 3\alpha_2\right) - (1-\varepsilon)\,(\alpha_1 + \alpha_2)\sum_{i=1}^{2}\alpha_i b_{i+1}^g + (4-\varepsilon)\,\alpha_1\alpha_2 x_3}{\alpha_2\left((2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\, 2\alpha_1\right)} \tag{15}$$

which we replace in the expression of $b_2^{\mathcal{N}}$ to obtain,

$$b_2^{\mathcal{N}} = \frac{(2+\varepsilon)\,\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^{g} - (1-\varepsilon)(\alpha_1-\alpha_2)\,x_1}{(2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\,2\alpha_1} \tag{16}$$

To find the value of the threshold δ simply find the value of ε that solves,

$$\frac{\left((2+\varepsilon)\,\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^{g} - (1-\varepsilon)(\alpha_1-\alpha_2)\,x_1\right)}{(2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\,2\alpha_1} - \frac{\left(3\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}^{g}\right)}{3\alpha_2 - (1-\varepsilon)\,\alpha_1} = 0$$

and

$$\frac{\left(x_1(\alpha_1-\alpha_2)\left((1-\varepsilon)\,\alpha_1 - 3\alpha_2\right) - (1-\varepsilon)(\alpha_1+\alpha_2)\sum_{i=1}^{2}\alpha_i b_{i+1}^{g} + (4-\varepsilon)\,\alpha_1\alpha_2 x_3\right)}{\alpha_2\left((2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\,2\alpha_1\right)} = 0$$

this gives,

$$\varepsilon^* = \delta = \frac{\left(1+\frac{\alpha_2}{\alpha_1}\right)\sum_{i=1}^{2}\alpha_i b_{i+1}^{g} - \left(1-\frac{\alpha_2}{\alpha_1}\right)(\alpha_1-3\alpha_2)\,x_1 - 4\alpha_2 x_3}{\left(1+\frac{\alpha_2}{\alpha_1}\right)\sum_{i=1}^{2}\alpha_i b_{i+1}^{g} - \left(1-\frac{\alpha_2}{\alpha_1}\right)\alpha_1 x_1 - \alpha_2 x_3}$$

Proof of the monotonicity and the decrease in $\boldsymbol{b}$ is straightforward. Denote by $\underline{b}_i^{\mathcal{N}}$ and $\bar{b}_i^{\mathcal{N}}$ the respective collusive bids for $\varepsilon \leq \delta$ and $\varepsilon > \delta$. $\forall \varepsilon \leq \delta$ the derivative of $b_2^{\mathcal{N}}$ with respect to ε is given by,

$$\frac{\partial\left(\underline{b}_2^{\mathcal{N}}\right)}{\partial\varepsilon} = \frac{3\alpha_2\left(\alpha_1 b_2^{g} + \alpha_2 b_3^{g} - \alpha_1 x_3\right)}{\left(3\alpha_2 - (1-\varepsilon)\,\alpha_1\right)^2} \geq 0$$

and the derivative with respect to $R^g = (\alpha_1 b_2^g + \alpha_2 b_3^g)$ by,

$$\frac{\partial\left(\underline{b}_2^{\mathcal{N}}\right)}{\partial R^g} = -\frac{1-\varepsilon}{3\alpha_2 - \alpha_1(1-\varepsilon)}$$

which is negative if $3\alpha_2 - \alpha_1(1-\varepsilon) \geq 0$ that is if $\frac{\alpha_2}{\alpha_1} \geq \frac{1-\varepsilon}{3}$ which is satisfied by the restriction that $b_2^{\mathcal{N}} > 0$. Now, $\forall \varepsilon > \delta$ the derivatives are given by:

$$\begin{aligned}
\frac{\partial\left(\bar{b}_2^{\mathcal{N}}\right)}{\partial\varepsilon} &= \frac{3\alpha_2\left(\alpha_1 b_2^{g} + \alpha_2 b_3^{g} + (\alpha_1-\alpha_2)\,x_1 - 2\alpha_1 x_3\right)}{\left(\alpha_2(2+\varepsilon) - 2\alpha_1(1-\varepsilon)\right)^2} \geq 0\\
\frac{\partial\left(\bar{b}_3^{\mathcal{N}}\right)}{\partial\varepsilon} &= \frac{3(\alpha_1+\alpha_2)\left(\alpha_1 b_2^{g} + \alpha_2 b_3^{g} + (\alpha_1-\alpha_2)\,x_1 - 2\alpha_1 x_3\right)}{\left(\alpha_2(2+\varepsilon) - 2\alpha_1(1-\varepsilon)\right)^2} \geq 0\\
\frac{\partial\left(\bar{b}_2^{\mathcal{N}}\right)}{\partial R^g} &= \frac{-(1-\varepsilon)}{(2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\,2\alpha_1} \leq 0\\
\frac{\partial\left(\bar{b}_3^{\mathcal{N}}\right)}{\partial R^g} &= \frac{-(1-\varepsilon)(\alpha_1+\alpha_2)}{\alpha_2\left((2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\,2\alpha_1\right)} \leq 0
\end{aligned}$$

in which the last two denominators present positive signs if $\theta \geq 2\frac{(1-\varepsilon)}{(2+\varepsilon)}$.

Proof of proposition 3

The first and second derivative of $\Gamma'_{\mathcal{N}}$ are respectively equal to:

$$\frac{\partial}{\partial\varepsilon}\left(\Gamma'_{\mathcal{N}}\right) = 3\alpha_2\left(3\alpha_2-\alpha_1\right)\frac{R^g-\alpha_1 x_3}{\left(3\alpha_2-\left(1-\varepsilon\right)\alpha_1\right)^2}$$

which is positive if $3\alpha_2 \geq \alpha_1$ and

$$\frac{\partial^2}{\partial\varepsilon^2}\left(\Gamma'_{\mathcal{N}}\right) = 6\alpha_1\alpha_2\left(\alpha_1-3\alpha_2\right)\frac{R^g-\alpha_1 x_3}{\left(3\alpha_2-\left(1-\varepsilon\right)\alpha_1\right)^3}$$

which presents a negative sign whenever $3\alpha_2 \geq \alpha_1$. Thus, in the domain $[0,\delta^*)$ the *bidding agency*'s profit function is concave. The function is non-differentiable in $\varepsilon = \delta^*$ and the first and second derivatives of $\Gamma''_{\mathcal{N}}$ are, respectively, given by:

$$\frac{\partial}{\partial\varepsilon}\left(\Gamma''_{\mathcal{N}}\right) = -6\alpha_2\left(\alpha_1-\alpha_2\right)\frac{R^g+\left(\alpha_1-\alpha_2\right)x_1-2\alpha_1 x_3}{\left(\left(2+\varepsilon\right)\alpha_2-\left(1-\varepsilon\right)2\alpha_1\right)^2}$$

which presents a negative sign and

$$\frac{\partial^2}{\partial\varepsilon^2}\left(\Gamma''_{\mathcal{N}}\right) = 12\left(2\alpha_1+\alpha_2\right)\left(\alpha_1-\alpha_2\right)\frac{R^g+\left(\alpha_1-\alpha_2\right)x_1-2\alpha_1 x_3}{\left(\left(2+\varepsilon\right)\alpha_2-\left(1-\varepsilon\right)2\alpha_1\right)^3}$$

which presents a positive sign. Thus, the profit function is a convex function over the domain $(\delta^*,1)$.

Proof of proposition 5

In order for player 2 to be indifferent to winning the second position at price $b_3^{\mathcal{N}}$ or winning top position at price $b_2^{\mathcal{N}}$, the following relation should be satisfied:

$$\alpha_2\left(x_2-b_3^{\mathcal{N}}\right)-\omega_2+\frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} = \alpha_1\left(x_2-b_2^{\mathcal{N}}\right)-\omega_2$$

For player three to be indifferent to being assigned to the second position or not being assigned at all under the collusive agreement it should be that $\frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} = \alpha_2\left(x_3-b_3^{\mathcal{N}}\right)$. Rearranging both relations, we obtain the following pair of equations:

$$\alpha_1 b_2^{\mathcal{N}} = \frac{3}{2+\varepsilon}\left(x_2\left(\alpha_1-\alpha_2\right)\right)+\frac{4-\varepsilon}{2+\varepsilon}\alpha_2 b_3^{\mathcal{N}}-\frac{1-\varepsilon}{2+\varepsilon}\left(\sum_{i=1}^{2}P_i^v\right) \tag{17}$$

$$\alpha_2 b_3^{\mathcal{N}} = \frac{3}{2+\varepsilon}\alpha_2 x_3+\frac{1-\varepsilon}{2+\varepsilon}\left(\alpha_1 b_2^{\mathcal{N}}-\sum_{i=1}^{2}P_i^v\right) \tag{18}$$

Recall that $P_i^v = \sum_{k=i+1}^{m+1} x_k(\alpha_{k-1}-\alpha_k)$. Rearranging terms, plugging (18) into (17) and using equation (8), we obtain $\alpha_1 b_2^{\mathcal{N}} = x_2\left(\alpha_1-\alpha_2\right)+\alpha_2 x_3 = P_1^v$ and $\alpha_2 b_3^{\mathcal{N}} = \alpha_2 x_3 = P_2^v$. The solution corresponds for any $\varepsilon \in [0,1]$ to the same equilibrium bids and payments of the *VCG*-equivalent equilibrium bids profile of equation (8) and thus results in the same outcome as the *symmetric non-cooperative* equilibrium.

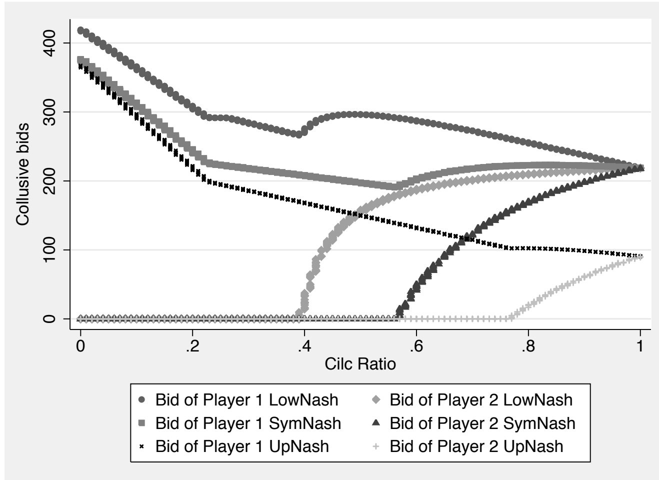


Figure 6: Equilibrium collusive bids when $\varepsilon = 0$ against θ respectively with outside bid profile $\boldsymbol{b}^l, \boldsymbol{b}^v, \boldsymbol{b}^u$ when $H(\alpha) \sim \mathcal{E}_{(0,23)}$.

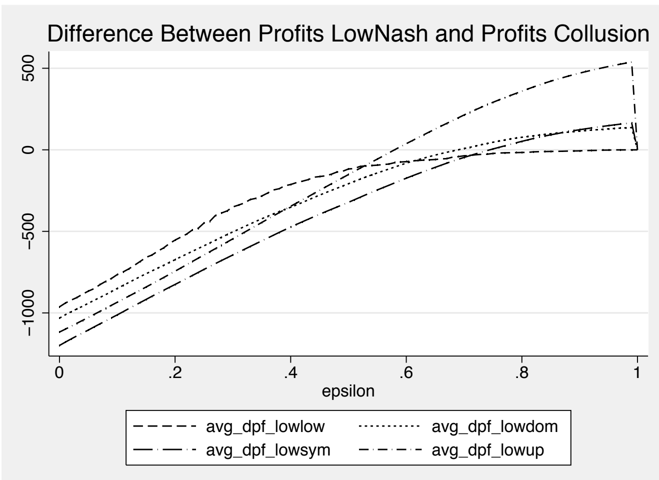
B Experimentations

We run 1000 instance of the one-shot *GSP* game, using the General Algebraic Modeling System (GAMS), in which, following Cary et al. (2008), each valuation is drawn from a distribution $G(x) \sim \mathcal{N}(500, 200)$ setting the value of X_1, X_2 and X_3 respectively to $x_1 = 592.7$, $x_2 = 565.535$ and $x_3 = 437.331$. On average, we observe a *ctr* of 0.23% on higher positions which allows us to reasonably set $\lambda = 0.23$ giving $E(\alpha) = 4.3$ (Synodiance 2013 synodiance.ctr.study2013) thus $H(\alpha) \sim \mathcal{E}_{(0.23)}$. The exponential law generates numbers lying between 0 and 1 which can be directly interpreted as click probability or click rates. For each instance, we computed the collusive bids resulting from the *agency*'s maximisation problem for each $\varepsilon \in [0, 1]$. The outside option $\pi_i(b_i, \boldsymbol{b}_{-i}, x_i)$ is *ex-ante* determined according to $\boldsymbol{b}^l, \boldsymbol{b}^d, \boldsymbol{b}^v, \boldsymbol{b}^u$. The bid vector $\boldsymbol{b}^l$ involves each bidder playing at the lower bound of equation (7), $\boldsymbol{b}^d$ is simply $\boldsymbol{b}^l$ with the restriction that bidder 3 plays truthfully, $\boldsymbol{b}^v$ are the bids sustaining the *VCG-equivalent* outcome defined by the bid vector in equation (8) and $\boldsymbol{b}^u$ corresponds to the upper bound of equation (7).

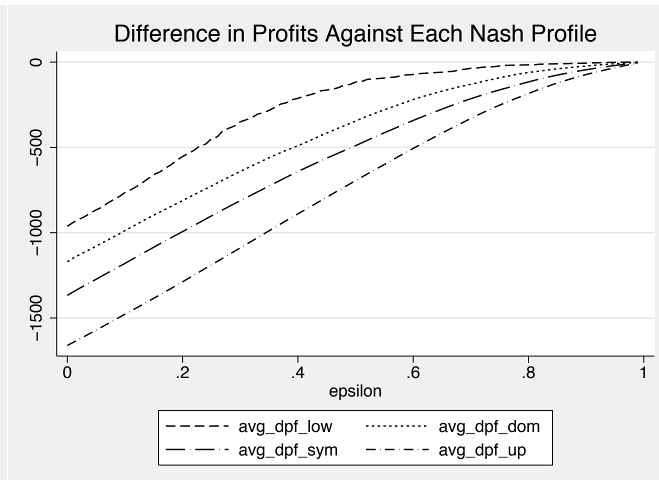
B.1 Bids, profits and revenues

Figure 6 depicts the positive relation existing between the ratio of clicks, we denoted by θ, when $\varepsilon = 0$ and the cooperative bids of players 1 and 2. In such a situation, the rivalry from the lowest advertiser is muted and the collusive outcome is maximised. This figure shows the underlying idea of proposition 2 that the degree of difference in clicks between both positions has to be low enough to induce an efficient allocation. That is, to avoid a random assignment between the lowest-valuing member and the second-highest valuing one.

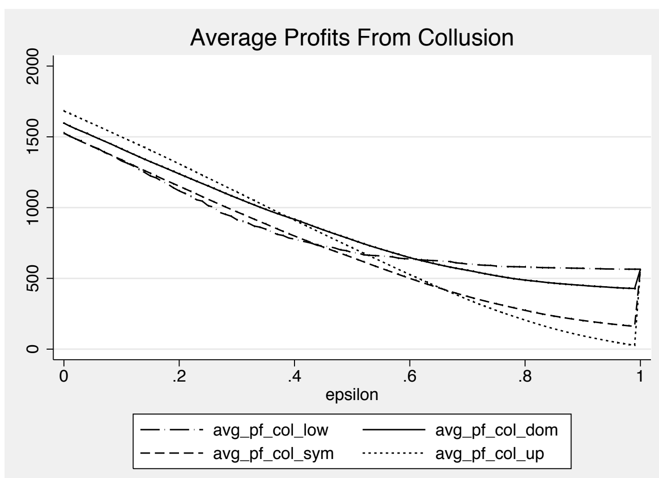
Figure 7 expresses the underlying idea of corollary 1. That is, all individual surpluses are lower-bounded by the individual surpluses produced in the game sustained by the corresponding *non-cooperative* price vector. We represented the average difference between collusive surpluses when contributions are computed according to each outside option price, i.e., $p_k^g = b_{k+1}^g$ with $g = l, d, v, u$, and: (i) the *non-cooperative* surplus produced by bid vector $\boldsymbol{b}^l$ and (ii) the surplus that individuals would have in each corresponding *non-cooperative* equilibrium. Let $\psi_{(.)}$ be the average difference for each case. We computed



(a) Difference in profits with profits in *Low*-Nash

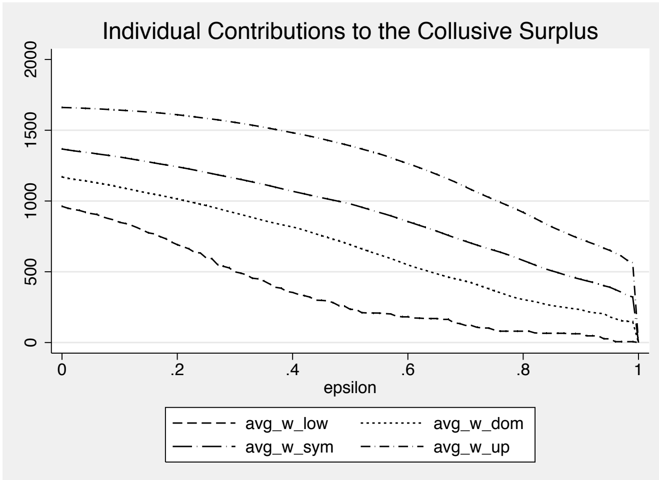


(b) Difference in profits with each $\boldsymbol{b}^g$

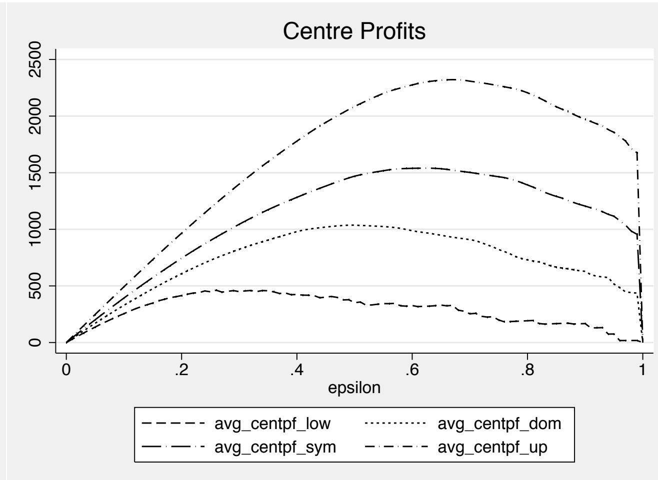


(c) Joint profits from collusion

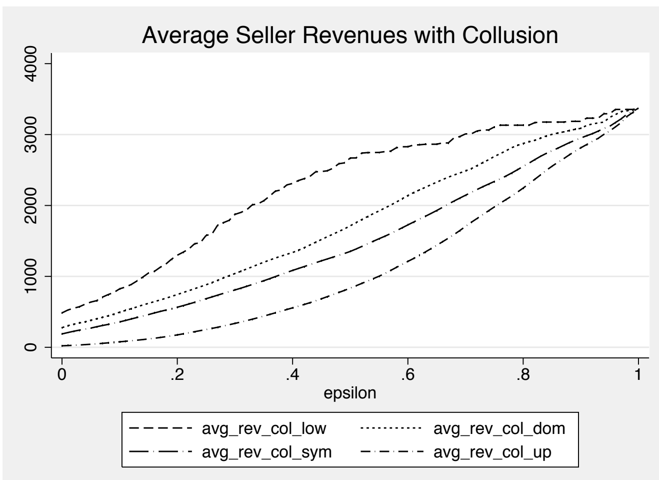
Figure 7: Average difference in surplus with the bids sustaining $\boldsymbol{b}^l$, in payoffs against each outside option and joint profits when the *agency* set a fee $\varepsilon \geq 0$.



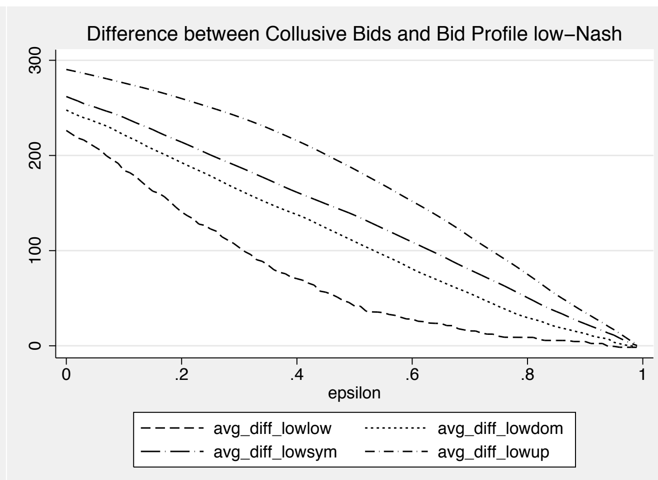
(a) Individual Contributions



(b) Revenues of the *agency*



(c) Seller's revenues



(d) Difference in bids

Figure 8: Average seller's revenues, individual contributions, revenues of the bidding agency according to outside options $\boldsymbol{b}^l, \boldsymbol{b}^d, \boldsymbol{b}^v$ and $\boldsymbol{b}^u$ and difference between bid profile $\boldsymbol{b}^l$ and each collusive bid profile.

$\psi_{(i)} = \pi_k \left(p_k^l\right) - \pi_k^{\mathcal{N}} \left(p_k^g\right)$ and $\psi_{(ii)} = \pi_k \left(p_k^g\right) - \pi_k^{\mathcal{N}} \left(p_k^g\right)$. We can observe that $\psi_{(ii)} \mapsto 0$ as $\varepsilon \mapsto 1^-$ which makes sense as both individual contributions tend to zero and each individual rationality constraint is binding as the collusive bids increase with ε. Although for high enough values of ε, the bid profile computes according to the outside option $\boldsymbol{b}^l$ strictly dominates all other collusive profiles from the bidders' viewpoint. Note the disruption at $\varepsilon = 1$. Joint profits jump to the joint surplus achieved by $\boldsymbol{b}^l$ (Figures 7a and 7c).

Figure 8 depicts the idea of proposition 4 that the revenue of the seller is upper-bounded by the revenue produced under price vector $\boldsymbol{b}^l$. If the collusive mechanism is computed according to the outside option $\boldsymbol{b}^l$, $\boldsymbol{b}^d$, $\boldsymbol{b}^v$ or $\boldsymbol{b}^u$, we can observe in Figure 8d, that as $\varepsilon \mapsto 1$ collusive prices converge to the level sustaining the *non-cooperative* outcome $\boldsymbol{b}^l$. As a result, revenues for the seller also converge to the same revenue level (around 3372) as shown in Figure 8c.

	Benevolent *agency*									
	Non-Cooperative Play					**Cartel**				
	$\boldsymbol{b}^l$	$\boldsymbol{b}^d$	$\boldsymbol{b}^v$	$\boldsymbol{b}^u$	Mean	$\boldsymbol{b}^l$	$\boldsymbol{b}^d$	$\boldsymbol{b}^v$	$\boldsymbol{b}^u$	Mean
Surplus										
Player 1	1024.32	1024.32	466.81	60.98	644.11	2039.84	2231.134	1878.852	1756.32	1976.53
Player 2	700.97	287.65	287.65	0	319.06	1716	1494.49	1699.67	1695.34	1651.52
Player 3	.	.	.	.	.	1015.54	1206.84	1412.02	1695.34	1332.43
Contributions										
Player 1	.	.	.	.	.	2478.76	2639.27	3254.81	3817.13	3047.5
Player 2	.	.	.	.	.	567.86	981.24	981.24	1268.9	949.8
Player 3	.	.	.	.	.	0	0	0	0	
Average Contributions	.	.	.	.	.	1015.542	1206.84	1412.02	1332.43	
Seller Revenue	3450.82	3864.2	4421.7	5115.17	4212.9	404.2	243.7	185.64	29.15	215.67
Total Surplus	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15
Cartel Surplus	.	.	.	.	.	4771.95	4932.46	4990.5	5147	4960.48
Bids										
Player 1	453.53	514.83	592.71	592.71	538.44	304.38	274.31	244.81	194.88	254.6
Player 2	437.3	437.3	514.83	581.96	492.86	78.89	49.9	38.98	7.04	43.73
Player 3	170.14	437.3	437.3	565.54	402.59	0	0	0	0	0
Average Bids	353.67	463.16	514.96	580.07		127.76	108.1	94.6	67.31	

Table 1: Equilibrium outcomes with a benevolent *agency* for $H(\alpha) \sim \mathcal{E}_{(0.23)}$ and $G(x) \sim \mathcal{N}_{(500,200)}$

B.2 Comparison between the benchmark and the collusive outcome and thresholds for ε

Table 1 describes the collusive outcome implemented by a *bidding agency* that acts as a benevolent agent, i.e. $\varepsilon = 0$ and the *non-cooperative* outcome corresponding to each Nash extremum we consider. Conversely, Tables 3 and 4 describe the solution implemented by the mechanism when the *agency* imposes a flat-fee $\varepsilon > 0$ on collusive gains and the corresponding *non-cooperative* outcomes, again for each Nash extremum. We represent each individual profit, contribution, seller's revenue, joint profit (cartel surplus) and individual bid.

Description of the non-cooperative equilibria and of the collusive mechanism with a benevolent *bidding agency* from Table 1: If each player behaves according to the lower bound of relations (7), i.e., $\boldsymbol{b}^l$, then equilibrium bids involve bid shading from each player with respect to their valuations ($b_1 = 453.53 < 592, b_2 = 437.3 < 565.5$ and $b_3 = 170.1 < 437.3$), the final allocation is efficient as player 1 obtains the first position and player 2 the second (however it is not *locally-envy free*) and produces a total revenue of $R_u = 3450.8$ for the seller. Notice that this outcome strictly dominates the other three in terms of bidders' surplus. It transpires that if we consider the *bidding agency* as endorsing the role of a mediating device , when setting $\varepsilon = 1$, which is an equivalent situation as collusive profits strictly equate *non-cooperative* profits, the *agency* implements a Nash outcome that Pareto-dominates any other *non-cooperative* outcome from the bidders' viewpoint.

If the *agency* were to set $\varepsilon = 0$, i.e., no surplus extraction from the collusion, then internal rivalry is reduced to the lowest level compatible with an efficient allocation and the mechanism implements the

Threshold δ^*

	$\boldsymbol{b}^l$	$\boldsymbol{b}^d$	$\boldsymbol{b}^v$	$\boldsymbol{b}^u$
δ^*	0.143	0.241	0.318	0.516

Table 2: Threshold δ^* according to $\boldsymbol{b}^l, \boldsymbol{b}^d, \boldsymbol{b}^v, \boldsymbol{b}^u$

Corresponding NC outcomes

	$\boldsymbol{b}^l$	$\boldsymbol{b}^d$	$\boldsymbol{b}^v$	$\boldsymbol{b}^u$	Average
Surplus					
Player 1	790.28	790.28	633.53	104.9	579.77
Player 2	685.22	495.27	495.27	0	418.94
Player 3	.	.	.	.	.
Seller Revenue	3723.88	3913.85	4070.6	5094.43	4200.69
Total Surplus	5199.3	5199.3	5199.3	5199.3	5199.3
Bids					
Player 1	431.12	466.65	592.71	592.71	520.79
Player 2	437.3	437.3	466.65	571.75	478.25
Player 3	385.58	437.3	437.3	565.54	456.45
Average Bids	418	447.1	498.9	576.67	

Table 3: Equilibrium *NC* outcomes corresponding to the non-neutral *agency* case for $H(\alpha) \sim \mathcal{E}_{(0.23)}$ and $G(x) \sim \mathcal{N}_{(500,200)}$

first-best outcome as defined in definition 2. Only the highest and the second-highest valuing member are active at the main auction whereas the lowest-valuing member is inactive. For instance, take the solution implemented according to $\boldsymbol{b}^u$. Bidder 1 bids $b_1^{\mathcal{N}} \simeq 195$, bidder 2 bids $b_2^{\mathcal{N}} \simeq 7.04$ and bidder 3 does not bid. Although it does not affect allocative efficiency, the active collusion destroys the seller's revenues as they fall from $R^u = 5115.17$ to $R_u^{\mathcal{N}} = 29.15$ if bidders behave according to $\boldsymbol{b}^u$, which would also correspond to the best solution in terms of the bidders' welfare, i.e., $BS_u^{\mathcal{N}} = 5147$.

Description of Tables 3 and 4: The computed values of the optimal fee threshold δ^* are presented for each outside option in Table 2. We take the overall average value of the 1000 instances run. We compute in Table 4 the collusive solution when the *agency* set $\varepsilon > 0$ for which data were arbitrarily divided in three parties. Results for the optimum situation $\varepsilon^* = \delta^*$ are depicted in the middle class of data, the left-hand side of the table corresponds to a flat-fee $\varepsilon' = \frac{\delta^*}{2}$ and the right hand side to a flat-fee $\varepsilon'' = 1.5\delta^*$. In contrast to the benevolent situation, the *agency*, by setting the flat-fee to its optimal level, implements an outcome that, from the bidders' viewpoint, strictly Pareto-dominates all other collusive solutions, i.e., the solution computed according to $\boldsymbol{b}^l$ for which individual surpluses are $\pi_1^{\mathcal{N}} \simeq 1448.38$, $\pi_2^{\mathcal{N}} \simeq 1343.4$ and $\pi_3^{\mathcal{N}} \simeq 658$.

It can be observed that the *bidding agency*'s profits are maximum when $\varepsilon^* = \delta^*$ and strictly increase whenever the outside option is assumed to be that of high equilibrium prices, in contrast to the bidders' individual surpluses. Once $\varepsilon > \delta^*$, these profits mechanically fall as collusive bids strictly increase causing contributions to fall. This corresponds to the underlying idea of proposition 3. For instance, the solution implemented according to the outside option $\boldsymbol{b}^l$ gives $\Gamma_{\mathcal{N}} \simeq 201.31$, an average contribution $\bar{\omega} \simeq 831.77$ when $\varepsilon' = 0.5 * \delta^*$. This gives $\Gamma_{\mathcal{N}} \simeq 366.8$, an average contribution $\bar{\omega} \simeq 780.37$ when $\varepsilon' = \delta^*$. This gives $\Gamma_{\mathcal{N}} \simeq 0$, the average contribution $\bar{\omega} \simeq 0$ when $\varepsilon' = 1.5 * \delta^*$. The seller strictly benefits from an increase in ε and, note, its revenues are quite close to their corresponding *non-cooperative* level (around $R^{\mathcal{N}} \simeq 3723$). Once the fee level crosses the threshold δ^*, collusive bids strictly converge to the lowest Nash equilibrium in price and the lowest-valuing member competition can no longer be constricted.

Active *agency*

Fee	$\varepsilon' = 0.5\delta^*$					$\varepsilon = \delta^*$					$\varepsilon'' = 1.5\delta^*$				
	$\boldsymbol{b}^l$	$\boldsymbol{b}^d$	$\boldsymbol{b}^v$	$\boldsymbol{b}^u$	Mean	$\boldsymbol{b}^l$	$\boldsymbol{b}^d$	$\boldsymbol{b}^v$	$\boldsymbol{b}^u$	Mean	$\boldsymbol{b}^l$	$\boldsymbol{b}^d$	$\boldsymbol{b}^v$	$\boldsymbol{b}^u$	Mean
Threshold	0.085	0.115	0.136	0.236	.	0.169	0.23	0.271	0.472	.	0.254	0.345	0.407	0.709	.
Surplus															
Player 1	1554.94	1604.94	1487.98	1203.46	1462.83	1448.38	1448.38	1291.63	763.1	1237.87	790.272	984.68	867.175	324.98	741.77
Player 2	1449.94	1309.97	1349.76	1098.47	1302.03	1343.4	1153.41	1153.41	658.10	1077.07	685.28	689.71	728.85	219.98	580.98
Player 3	764.66	814.67	854.46	1098.47	883.06	658.1	658.1	658.1	658.1	658.1	0	194.41	233.65	219.98	162.01
Agency Payoff	201.31	314.73	410.47	1025.63	488.03	366.8	556.77	713.52	1737.35	843.61	0	287.91	458.4	1455.03	550.33
Seller Revenue	1228.58	1155.12	1096.76	773.41	1063.47	1382.78	1382.78	1382.78	1382.78	1382.78	3723.9	3042.72	2911.25	2979.45	3164.33
Total Surplus	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43
Cartel Surplus	3769.54	3729.58	3692.2	3400.4	3647.93	3449.86	3259.88	3103.14	2079.31	2973.05	1475.55	1868.8	1829.78	764.95	1484.8
Contributions															
Player 1	995.7	1069.15	1284.28	2136.15	1371.34	841.5	841.5	998.25	1526.78	1052	0	243.38	446.24	947.5	409.23
Player 2	1499.6	1689.58	1689.58	2184.9	1765.9	1499.6	1689.58	1689.58	2184.9	1765.9	0	627.76	713.1	1167.49	627.1
Player 3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
Average	831.77	919.58	991.28	1440.34	.	780.37	843.69	895.94	1237.22	.	0	290.38	386.45	705	.
Bids															
Player 1	283.71	274.37	266.93	218.38	260.85	303.73	303.73	303.73	303.73	303.73	437.3	397.39	389.06	390.81	403.65
Player 2	240.38	226.73	215.9	152.52	208.88	269.17	269.17	269.17	269.17	269.17	437.3	388.1	378.43	382.65	396.63
Player 3	.	.	.	.	.	.	.	.	.	.	385.58	274.14	253.07	265.43	294.56
Average Bids	174.7	167.03	160.94	123.63	.	190.97	190.97	190.97	190.97	190.97	420.1	353.21	340.18	346.3	.

Table 4: Collusive outcomes when the *agency* sets $\varepsilon = \{0.5\delta^*, \delta^*, 1.5\delta^*\}$ for $H(\alpha) \sim \mathcal{E}_{(0.23)}$ and $G(x) \sim \mathcal{N}_{(500,200)}$

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